

Superconformal field theory and Jack superpolynomials

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ABSTRACT: We uncover a deep connection between the $\mathcal{N} = 1$ superconformal field theory in $2D$ and eigenfunctions of the supersymmetric Sutherland model known as Jack superpolynomials (sJacks). Specifically, the singular vector at level $rs/2$ of the Kac module labeled by the two integers r and s are given explicitly as a sum of sJacks whose indexing diagrams are contained in a rectangle with r columns and s rows. As a second compelling evidence for the distinguished status of the sJack-basis in SCFT, we find that the degenerate Whittaker vectors (Gaiotto states) can be expressed as a remarkably simple linear combination of sJacks. As a consequence, we are able to reformulate the supersymmetric version of the (degenerate) AGT conjecture in terms of the combinatorics of sJacks. Note that the closed-form formulas for the singular vectors and the degenerate Whittaker vectors, although only conjectured in general, have been heavily tested (in some cases, up to level $33/2$). Note also that both the Neveu-Schwarz and Ramond sectors are treated.

KEYWORDS: Super Conformal Field Theory, singular vectors, Whittaker vectors, symmetric polynomials, Calogero-Moser-Sutherland models

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1 Introduction

The Jack polynomials provide representations of the states in Virasoro highest-weight modules. This statement is to be understood in the context of the Feigin-Fuchs representation and the fact that free-bosonic modes can be represented by power-sum functions and their derivatives (see for instance [41]). In consequence, every state in the Fock space is associated to a symmetric polynomial and any basis of symmetric polynomials such as the Jack polynomials can be used to describe a given state.

But the Jack-polynomial basis displays remarkable features. It was indeed observed in [37] – and later in [3] – that the singular vector in a Kac module can be represented by a single Jack polynomial. Explicitly, if the central charge is $c = 13 - 6(t + t^{-1})$ then the singular vector $|\chi_{r,s}\rangle^{\text{Vir}}$ at level rs in the Kac module with highest-weight state $|h_{r,s}(t)\rangle^{\text{Vir}}$ (cf. eq. (A.6)) is associated to the Jack with parameter $\alpha = t$ and indexed by an $s \times r$ rectangular diagram: $|\chi_{r,s}\rangle^{\text{Vir}} \longleftrightarrow P_{(rs)}^{(\alpha)}$. This relation clearly singularizes the Jack basis. Recent works on the AGT conjecture [2] have also pinpointed the Jack polynomials basis [4, 38, 48], a point that is particularly clear in theories whose underlying algebra is the Virasoro times a $u(1)$ algebra [1].

The aim of this work is to show that the role played by the Jacks in CFT is performed by their $\mathcal{N} = 1$ supersymmetric version in superconformal field theory (SCFT).

The Jack polynomials are eigenfunctions of the Sutherland Hamiltonian, which is the trigonometric version of the Calogero-Moser-Sutherland Hamiltonian (up the contribution of the ground-state wavefunction). They depend upon a parameter α which is nothing but the inverse of the Sutherland coupling constant β .¹

The Jack superpolynomials [16, 17] (or sJacks for short – also known as Jack polynomials in superspace) are natural extensions of the usual Jack polynomials in that they are eigenfunctions of the supersymmetric generalization of the Sutherland Hamiltonian [13, 42]. As such, they depend upon both commuting and anticommuting variables.

We first show, via a free-field representation of the super-Virasoro algebra, that any state in SCFT highest-weight modules can be represented by a linear combination of sJacks. But at this level, the sJacks are not singularized yet: a state can be represented by a combination of any family of symmetric superpolynomials. The bottom line of this correspondence is that a state can be associated with a superpartition.

The truly striking property that distinguishes the sJack-basis is that the singular vectors in SCFT can be represented by simple combinations of sJacks. And this representation holds true for both the Neveu-Schwarz (NS) and the Ramond (R) sectors. To illustrate the drastic simplification that our new singular-vector representation provides, let us note that the sJack-representation of the NS singular

¹The relation between CFT and the Sutherland model is rich and multifaceted. Apart from their previously mentioned connections concerning the singular vectors and the AGT conjecture, they are related through the Schramm-Loewner Evolution (SLE) (see for instance [15, 23]) and the fractional quantum Hall effect (FQHE) [10, 26, 27].

vector at level $33/2$ contains only 11 terms, while its expression in terms of the modes L_n and G_r is expected to contain as many terms as the number of independent vectors in the Verma module at level $33/2$, which is 1687.

The particular combination of sJacks that appear in the expression of a singular vector is controlled by a diagram whose shape is a rectangle. But in the supersymmetric context, the rectangle-rule is more intricate than in the usual case where the rectangle selects a single Jack. To give an idea of the rule, we must recall that, in the same way as a Jack is labeled by a partition, an sJack is labeled by a superpartition. A superpartition is characterized by two ordinary partitions. When glued together in a certain way, the diagrams associated to these two partitions need to fill the rectangle with r columns and s rows for the labels corresponding to the singular vector $|\chi_{r,s}\rangle$ at level $rs/2$. We call such superpartitions self-complementary.

The self-complementarity specifies those terms that appear in the representation of a given singular vector. But what is rather astonishing – and a priori totally unexpected – is that we can also obtain all the coefficients in the linear combination of sJacks that represents the singular vector. In other words, using the sJack-basis we end up with a fully explicit conjectural expression for all the singular vectors in SCFT. The general formula is given in eq (5.13).

It is worth mentioning that there are two infinite sequences of singular vectors that do contain a single representative sJack, namely those for which r or s is equal to 1 (see eqs (4.21) and (4.22)). Interestingly, these are precisely the single instances for which an explicit mode-expression was known (see [9] and [47] for the NS and R sectors, respectively).

The recent work on the AGT conjecture has provided a novel CFT-intrinsic way of revealing the special status of the Jack-basis. Recall that in specializing the AGT conjecture to asymptotically free theories, Gaiotto [29] was led to the consideration of Whittaker vectors, namely vectors not annihilated by L_1 (in the degenerate or matter-free case) or not annihilated by L_1 nor L_2 (in the general case). In the degenerate case, an elegant expression for the Whittaker vector has been obtained in the Jack basis [48] (see also [4]). This result is generalized here (conjecturally) to the superconformal case. We thus end up with a beautiful closed-form polynomial representation for the SCFT degenerate Whittaker vector in both the NS and R sectors. We stress the remarkable simplicity of these expressions and at the same time their very non-trivial character in the sense that they are not immediate extensions of their Virasoro relatives. Again, we take this as a further confirmation of the relevance of the sJack basis in SCFT.

The article is concerned with the relationship between sJacks and the states in SCFT. But for those readers unfamiliar with the analogous connection – in particular, in relation with singular vectors – in the non-supersymmetric case, a brief summary is presented in Appendix A. The core of the article is organized as follows. In Section 2, the notions of highest-weight representations of the superconformal algebra and Kac modules are briefly reviewed. Then, by means of the free-field representation, a correspondence is made between the states in highest-weight modules and symmetric superpolynomials. The SCFT-distinguished family of symmetric superpolynomials, the sJacks, are introduced in Section 3. To complement the formal definition of these polynomials, some explicit expansion formulas are given in Appendix C. In Section 4, the lowest-level singular vectors are re-expressed in the sJacks basis: these linear combinations of sJacks are seen to be both simple, in the sense that they involve very few terms, and tractable, in that their coefficients factorize nicely. By inspection, we have obtained the full expression for three infinite series, corresponding to the $r = 1, 2, 3$ singular vectors. Next, it is shown that the sJacks duality-property allows to relate the (r, s) and (s, r) singular vectors (a transformation rule that thereby produces three new infinite series: $s = 1, 2, 3$). These special closed-form formulas are generalized in Section 5: there, the general expansion coefficient of a given

sJack contributing to the polynomial representation of the (r, s) singular vector is written in a fully explicit way. The efficiency of the sJack basic in SCFT is further demonstrated in Section 6, where are displayed the explicit sJack-representations of the degenerate Whittaker vector and its norm squared. This leads, in the conclusion, to a reformulation the supersymmetric analogue of the asymptotic AGT conjecture as a combinatorial conjecture involving scalar products of sJacks. To keep the presentation as simple as possible, only the NS sector is treated in the main text. All the results related to the Ramond sector are relegated to Appendix B.

2 Correspondence between SCFT and symmetric superpolynomials

In this section, we present the connection between symmetric superpolynomials and states in highest-weight modules over the super-Virasoro algebra, which is given by the following commutations [11, 28]

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0} \\ [L_n, G_k] &= \left(\frac{n}{2} - k\right)G_{n+k} \\ \{G_k, G_l\} &= 2L_{k+l} + \left(k^2 - \frac{1}{4}\right)\frac{c}{3}\delta_{k+l,0}, \end{aligned} \quad (2.1)$$

where as before, $n, m \in \mathbb{Z}$ and c denotes the central charge. In the NS sector, k and l belong to $\mathbb{Z} + 1/2$, while in the R sector, $k, l \in \mathbb{Z}$.

As in the Virasoro case (reviewed in Appendix A), this connection goes through the free-field representation of the conformal algebra. The relevant representation of the superconformal algebra is in terms of the modes of a free bosonic super-field, or equivalently, the modes of a free boson and a free fermion, a representation that is reviewed in Section 2.2 [11]. The link with symmetric superpolynomials is presented in Section 2.3, where the notion of superpartition is seen to arise naturally. But before presenting this key relationship, we review the notion of Kac modules. And, as mentioned in the introduction, in order to streamline the presentation, we confine ourself to the NS sector in the main text.

2.1 Kac modules

A highest-weight module over the NS sector is characterized by the central charge c and a highest weight h , and is spanned over \mathbb{C} by all states of the form

$$G_{-k_1} \cdots G_{-k_p} L_{-n_1} \cdots L_{-n_q} |h\rangle^{\text{NS}} \quad k_i, n_i > 0. \quad (2.2)$$

$|h\rangle^{\text{NS}}$ is the highest-weight state, characterized by the following three conditions:

$$L_0 |h\rangle^{\text{NS}} = h |h\rangle^{\text{NS}}, \quad L_n |h\rangle^{\text{NS}} = 0 \quad \forall n > 0 \quad \text{and} \quad G_k |h\rangle^{\text{NS}} = 0 \quad \forall k > 0. \quad (2.3)$$

Due to the third relation in (2.1), L_{2k} and L_{2k+1} can be respectively written as $G_k G_k$ and $\{G_k, G_{k+1}\}/2$, which implies that the second condition in (2.3) is a consequence of the third one. Simple manipulations with (2.1) show that requiring

$$L_0 |h\rangle^{\text{NS}} = h |h\rangle^{\text{NS}}, \quad G_{\frac{1}{2}} |h\rangle^{\text{NS}} = 0 \quad \text{and} \quad G_{\frac{3}{2}} |h\rangle^{\text{NS}} = 0 \quad (2.4)$$

suffices to establish the highest-weight nature of $|h\rangle^{\text{NS}}$ in the NS sector.

Let us consider a state $|\chi\rangle^{\text{NS}}$ at level k in the highest-weight module just described, i.e., $|\chi\rangle^{\text{NS}}$ is a linear combination of states of the form (2.2) such that $L_0|\chi\rangle^{\text{NS}} = (h+k)|\chi\rangle^{\text{NS}}$. We say that $|\chi\rangle$ is a singular vector (or null vector) if it behaves as a highest-weight state, meaning that it satisfies

$$G_{\frac{1}{2}}|\chi\rangle^{\text{NS}} = 0 \quad \text{and} \quad G_{\frac{3}{2}}|\chi\rangle^{\text{NS}} = 0. \quad (2.5)$$

As in the usual Virasoro case, there are special highest-weight states whose conformal dimension $h = h_{r,s}$ is parametrized by two positive integers (r, s) and is related to c via the auxiliary parameter t :

$$c = \frac{15}{2} - 3\left(t + \frac{1}{t}\right) \quad \text{and} \quad h_{r,s} = \frac{t}{8}(r^2 - 1) + \frac{1}{8t}(s^2 - 1) - \frac{1}{4}(rs - 1). \quad (2.6)$$

This dimension formula pertains to the NS sector, in which $r + s$ is even. For such so-called Kac modules, there is a singular vector at level $rs/2$.²

2.2 Free-field representation

We introduce the standard free bosonic and fermionic modes, denoted respectively a_n and b_r , which satisfy

$$[a_n, a_m] = n\delta_{n+m,0}, \quad [a_0, \pi_0] = 1, \quad \{b_k, b_l\} = \delta_{k+l,0}, \quad (2.7)$$

where π_0 is the canonical conjugate of the zero-mode a_0 . The indices n, m are integer, while k and l are half-integer. We use the standard notation $::$ for the normal ordering (the largest mode is placed at the right end, which introduces a sign if two fermionic modes are interchanged). It is a simple exercise to show that the following expressions provide a representation of the NS-sector superconformal algebra with central charge $c = 3/2 - 12\gamma^2$:

$$\begin{aligned} L_n &= -\gamma(n+1)a_n + \frac{1}{2} \sum_{m \in \mathbb{Z}} :a_m a_{n-m}: + \frac{1}{4} \sum_{k \in \mathbb{Z} + \frac{1}{2}} (n-2k) :b_k b_{n-k}: \\ G_k &= -2\gamma\left(k + \frac{1}{2}\right)b_k + \sum_{m \in \mathbb{Z}} a_m b_{k-m}. \end{aligned} \quad (2.8)$$

The Fock space \mathcal{F} to be considered here is associated to a free bosonic superfield. It is thus the tensor product of a free-boson Fock space and that of a free fermion. For instance, the highest-weight states built on the vacuum state

$$|0\rangle \equiv |0\rangle_{\text{bos}} \otimes |0\rangle_{\text{fer}}, \quad (2.9)$$

are given by the one-parameter family

$$|\eta\rangle \equiv |\eta\rangle_{\text{bos}} \otimes |0\rangle_{\text{fer}}, \quad \text{with} \quad |\eta\rangle_{\text{bos}} \equiv e^{\eta\pi_0}|0\rangle_{\text{bos}}, \quad (2.10)$$

Note that

$$a_n|0\rangle = b_k|0\rangle = 0, \quad \forall n, k > 0. \quad (2.11)$$

Using the latter equation and the commutation relation $[a_0, \pi_0] = 1$, it follows that

$$a_0|\eta\rangle = \eta|\eta\rangle. \quad (2.12)$$

²That there is at most one singular vector at each level is proved in [32] and the existence of a singular vector at level $rs/2$ is demonstrated in [30].

The full Fock space \mathcal{F} is the linear span over \mathbb{C} of all monomials

$$b_{-k_1} \cdots b_{-k_p} a_{-n_1} \cdots a_{-n_q} |\eta\rangle, \quad k_i, l_i > 0. \quad (2.13)$$

According to the representation (2.8), we have

$$L_0 |\eta\rangle = (\tfrac{1}{2}\eta^2 - \gamma\eta) |\eta\rangle, \quad (2.14)$$

which means that the conformal dimension of $|\eta\rangle$ is

$$h = \tfrac{1}{2}\eta^2 - \gamma\eta. \quad (2.15)$$

This last equation, together with $c = 3/2 - 12\gamma^2$ and the parametrization (2.6), finally allows us to identify the highest-weight state $|h_{r,s}\rangle^{\text{NS}}$ of the Kac module with the Fock space state $|\eta_{r,s}\rangle$, where

$$\eta_{r,s} = \frac{1}{2\sqrt{t}} \left((1+r)t - (1+s) \right). \quad (2.16)$$

Note that the correspondence also requires

$$\gamma = \eta_{0,0} = \frac{1}{2\sqrt{t}}(t-1). \quad (2.17)$$

2.3 Correspondence between states and the power-sum superpolynomials

We introduce the n -th power-sum in infinitely many variables, denoted p_n and its fermionic partner \tilde{p}_n defined as [18]

$$p_n = \sum_i x_i^n \quad (n > 0) \quad \text{and} \quad \tilde{p}_n = \sum_i \theta_i x_i^n, \quad (n \geq 0) \quad (2.18)$$

where $\theta_1, \theta_2, \dots$ are anticommuting variables. Note that \tilde{p}_0 is non-trivial, being equal to $\sum_i \theta_i$. Both p_n and \tilde{p}_n are instances of symmetric superpolynomials, that is, polynomials in the variables x_1, x_2, \dots and $\theta_1, \theta_2, \dots$ that remain unchanged under any simultaneous permutation of the form $(x_i, \theta_i) \leftrightarrow (x_j, \theta_j)$. An elementary result states that any element in the vector space over \mathbb{C} of all symmetric superpolynomials in infinitely many variables, denoted by \mathcal{R} , can be uniquely written as a polynomial in p_1, p_2, \dots and $\tilde{p}_0, \tilde{p}_1, \dots$ (see for instance [18]).

We are now in position to formulate a correspondence between the free-field modes and the differential operators acting on \mathcal{R} :

$$\begin{aligned} a_{-n} &\longleftrightarrow \frac{(-1)^{n-1}}{\sqrt{\alpha}} p_n & a_n &\longleftrightarrow n(-1)^{n-1} \sqrt{\alpha} \frac{\partial}{\partial p_n} \\ b_{-k} &\longleftrightarrow \frac{(-1)^{k-1/2}}{\sqrt{\alpha}} \tilde{p}_{k-1/2} & b_k &\longleftrightarrow (-1)^{k-1/2} \sqrt{\alpha} \frac{\partial}{\partial \tilde{p}_{k-1/2}} \end{aligned} \quad (2.19)$$

where $k, n > 0$ and α is a non-zero free parameter.³ The correspondence between \mathcal{F} and \mathcal{R} is given by

$$\begin{aligned} |\eta\rangle &\longleftrightarrow 1 \\ a_0 &\longleftrightarrow \eta \\ b_{-k_1} \cdots b_{-k_m} a_{-n_1} \cdots a_{-n_p} |\eta\rangle &\longleftrightarrow \zeta \tilde{p}_{k_1 - \frac{1}{2}} \cdots \tilde{p}_{k_m - \frac{1}{2}} p_{n_1} \cdots p_{n_p}, \end{aligned} \quad (2.20)$$

³Note that the first two relationships differ slightly from the corresponding ones in the Virasoro case given in (A.4).

with

$$k_i > k_{i+1} \geq \frac{1}{2} \quad \text{and} \quad n_i \geq n_{i+1} \geq 1, \quad (2.21)$$

and ζ is a constant described below.

The pattern of independent descendants states written on the lhs of (2.20) is easily justified: we can obviously order all the fermionic modes at the left and in decreasing values since they mutually anticommute and commute with the a_n 's. Relabeling the indices as

$$\begin{aligned} k_i - \frac{1}{2} &= \Lambda_i & \text{for } 1 \leq i \leq m, \\ n_i &= \Lambda_{i+m} & \text{for } 1 \leq i \leq p = \ell - m, \end{aligned} \quad (2.22)$$

the inequalities (2.21) become

$$\begin{aligned} \Lambda_i &> \Lambda_{i+1} \geq 0 & \text{for } 1 \leq i \leq m-1, \\ \Lambda_i &\geq \Lambda_{i+1} \geq 1 & \text{for } m+1 \leq i \leq \ell-1. \end{aligned} \quad (2.23)$$

Thus, any state in \mathcal{F} is in correspondence with a polynomial in \mathcal{R} indexed by two partitions: $\Lambda^a = (\Lambda_1, \dots, \Lambda_m)$, whose elements are strictly decreasing with $\Lambda_m \geq 0$, and $\Lambda^s = (\Lambda_{m+1}, \dots, \Lambda_\ell)$, which is a standard partition with non-zero elements. Together, the partitions Λ^a and Λ^s form the superpartition $\Lambda = (\Lambda_1, \dots, \Lambda_m; \Lambda_{m+1}, \dots, \Lambda_\ell)$ [16]. The non-negative integer m is called the fermionic degree of the superpartition Λ while its bosonic degree is given by $|\Lambda| = \sum_{i=1}^\ell \Lambda_i$. If $|\Lambda| = n$, we write $\deg(\Lambda) = (n|m)$. Let us also mention that ℓ is called the length of Λ .

Superpartitions are fundamental objects in the theory of symmetric superpolynomials [18–20]. Any element in a basis of the space \mathcal{R} is indexed by a superpartition. For instance, if f is a symmetric polynomial of degree n in the variables x_1, x_2, \dots and degree m in the variables $\theta_1, \theta_2, \dots$, then we can always write

$$f = \sum_{\Lambda} c_{\Lambda} p_{\Lambda} \quad (2.24)$$

where the sum extends over all superpartitions $\Lambda = (\Lambda_1, \dots, \Lambda_m; \Lambda_{m+1}, \dots, \Lambda_\ell)$ of degree $(n|m)$ and where p_{Λ} stands for the following product of power-sums:

$$p_{\Lambda} = \tilde{p}_{\Lambda_1} \cdots \tilde{p}_{\Lambda_m} p_{\Lambda_{m+1}} \cdots p_{\Lambda_\ell}. \quad (2.25)$$

Having made the connection (2.22) between modes and superpartitions, we easily see that the product of power-sums on the rhs of (2.20) is nothing but the polynomial p_{Λ} just defined in (2.25). Moreover, we are now able to express the constant $\zeta \equiv \zeta_{\Lambda}$ as

$$\zeta_{\Lambda} = \frac{(-1)^{|\Lambda|-\ell+m}}{\alpha^{\ell/2}} \quad (2.26)$$

Let us rewrite the state on the lhs of the third relation in (2.20) as

$$d_{-\Lambda} = d_{-\Lambda_1} \cdots d_{-\Lambda_\ell} \quad \text{with} \quad d_{-\Lambda_i} = \begin{cases} b_{-\Lambda_i - \frac{1}{2}} & \text{for } \Lambda_i \in \Lambda^a \\ a_{-\Lambda_i} & \text{for } \Lambda_i \in \Lambda^s. \end{cases} \quad (2.27)$$

In the Fock space, a generic term at level k is a linear combination of states $d_{-\Lambda} |\eta\rangle$ with $\text{level}(\Lambda) = k$, the level – the relative conformal dimension of the descendant state – being read off the correspondence

given in (2.27). Such states can thus be represented by linear combinations of the super-power-sum p_Λ of level k , i.e.,

$$\sum_{\text{level}(\Lambda)=k} c_\Lambda d_{-\Lambda} |\eta\rangle \longleftrightarrow \sum_{\text{level}(\Lambda)=k} \tilde{c}_\Lambda(\alpha) p_\Lambda \quad (2.28)$$

where $\tilde{c}_\Lambda(\alpha) = \zeta_\Lambda c_\Lambda$. The constraint on the sums ensures that all the terms have the same conformal dimension. When expressed in terms of the superpartition data, it reads:

$$\text{level}(\Lambda) = n + \frac{m}{2} \quad \text{if} \quad \deg(\Lambda) = (n|m) \quad (2.29)$$

An important observation in that regard is that, at a given level k , the value of m is not fixed: terms with different m can appear. However, the above relation forces the different values of m to differ by even integers, which guarantees the coherence of the equations with respect to their Bose-Fermi statistics.

Note that the sum on the rhs of (2.28) can be rewritten in terms of any basis of \mathcal{R} . In particular, it can be rewritten in terms of the Jack superpolynomials that will be introduced in the next section.

2.4 Differential representation of the super-Virasoro generators

The free field representation (2.8) and the correspondence (2.19) immediately imply that the generators G_k and L_n can be represented as differential operators acting on the space \mathcal{R} of symmetric superpolynomials. Let us denote these differential representations by \mathcal{G}_r and \mathcal{L}_n respectively. For instance, with the shorthand notation

$$\partial_n = \frac{\partial}{\partial p_n}, \quad \tilde{\partial}_n = \frac{\partial}{\partial \tilde{p}_n}, \quad \bar{\eta} = \sqrt{\alpha}\eta, \quad \bar{\gamma} = \sqrt{\alpha}\gamma \quad (2.30)$$

the $osp(1,2)$ -subalgebra generators are found to be represented by

$$\mathcal{G}_{1/2} = (\bar{\eta} - 2\bar{\gamma})\tilde{\partial}_0 + \sum_{n>0} (n\tilde{p}_{n-1}\partial_n - p_n\tilde{\partial}_n) \quad (2.31)$$

$$\mathcal{G}_{-1/2} = \frac{\bar{\eta}}{\alpha}\tilde{p}_0 + \sum_{n>0} (p_n\tilde{\partial}_{n-1} - n\tilde{p}_n\partial_n) \quad (2.32)$$

and

$$\begin{aligned} \mathcal{L}_1 &= (\bar{\eta} - 2\bar{\gamma})\partial_1 - \sum_{n>0} \left((n+1)p_n\partial_{n+1} + n\tilde{p}_{n-1}\tilde{\partial}_n \right) \\ \mathcal{L}_0 &= \frac{1}{2\alpha}\bar{\eta}(\bar{\eta} - 2\bar{\gamma}) + \sum_{n>0} n p_n \partial_n + \sum_{m\geq 0} \left(m + \frac{1}{2} \right) \tilde{p}_m \tilde{\partial}_m \\ \mathcal{L}_{-1} &= \frac{\bar{\eta}}{\alpha}p_1 - \sum_{n>0} n p_{n+1} \partial_n - \sum_{m\geq 0} (m+1) \tilde{p}_{m+1} \tilde{\partial}_m \end{aligned} \quad (2.33)$$

In the following, we will need mostly the expression of $\mathcal{G}_{\frac{1}{2}}$ and that of $\mathcal{G}_{\frac{3}{2}}$ given by

$$\mathcal{G}_{\frac{3}{2}} = -(\bar{\eta} + 2 - 2\alpha)\tilde{\partial}_1 + \alpha\partial_1\tilde{\partial}_0 - \sum_{n\geq 2} n\tilde{p}_{n-2}\partial_n + \sum_{n>0} p_n\tilde{\partial}_{n+1} \quad (2.34)$$

3 Jack superpolynomials

We have seen in the last section that any element of the superconformal algebra can be represented explicitly as a differential operator in the power-sums p_n and \tilde{p}_n . Moreover, to each state in the Kac module over the superconformal algebra, we can associate an element of the space \mathcal{R} of symmetric superpolynomials. Given that the set of all possible products of power-sums, p_Λ , constitute a basis for \mathcal{R} , any singular vector of the Kac module can be written as a linear combination of the form $\sum_\Lambda c_\Lambda p_\Lambda$. However, as will become clear in the next section, the power-sum basis is not convenient since it does not lead to simple expressions for the singular vectors. In the next subsections, we give a short introduction to the Jack superpolynomials (sJacks), which will prove to provide the best polynomial representation in SCFT.

Warning: As we will discuss shortly, the Jack superpolynomials are orthogonal with respect to a natural scalar product on the space of symmetric superpolynomials. However, it should be stressed that the superpolynomials constructed in [16] are not orthogonal. The correct orthogonal basis is introduced in [17] and from then on the terminology “Jack superpolynomials” has been reserved to the latter.

3.1 Supersymmetric Sutherland model and Jack superpolynomials

The Jack superpolynomials (sJacks) $P_\Lambda^{(\alpha)}$ [17] are natural extensions of the usual Jack polynomials (Jacks) in that they are eigenfunctions (with the contribution of the ground-state wavefunction factored out) of the Hamiltonian in the supersymmetric generalization of the Sutherland model (also known as the trigonometric Calogero-Moser-Sutherland model). The latter supersymmetric model [13, 16, 42] describes the interaction on the unit circle of N bosonic particles – represented by N commuting variables x_1, \dots, x_N – together with their respective fermionic partners – represented by N anticommuting variables $\theta_1, \dots, \theta_N$.

In [42], a set of N mutually commuting operators containing the Hamiltonian was found. However, in contrast with the non-supersymmetric case, these operators do not determine their common eigenfunctions uniquely, i.e., the eigenvalues are degenerate [16]. There is however a second set of bosonic conserved quantities, which disappear when all θ_i are set equal to 0. Let the first and second sets of commuting operators be respectively denoted by \mathcal{H}_n and \mathcal{I}_{n-1} , where $n = 1, \dots, N$ (the Hamiltonian \mathcal{H} of the supersymmetric model being equal to \mathcal{H}_2). The polynomial eigenfunctions of these operators depend upon x_i and θ_i : they are thus superpolynomials. In addition, they can be chosen to be symmetric, that is, invariant under the interchange of $(x_i, \theta_i) \leftrightarrow (x_j, \theta_j)$. Then, as demonstrated in [17], the Jacks $P_\Lambda^{(\alpha)}$ are the unique (up to a multiplicative factor) symmetric eigenfunctions of the whole set $\{\mathcal{H}_n, \mathcal{I}_{n-1}\}_{n=1}^N$.

As in the non-supersymmetric Sutherland model, where it suffices to consider only the analogue of \mathcal{H}_2 , the eigen-operator characterization can be formulated in terms of the lowest nontrivial member of each tower, namely \mathcal{H}_2 and \mathcal{I}_1 . After some transformations (involving the lower conserved charge \mathcal{H}_1), the two defining eigen-operators can be taken to be [20]:

$$D = \frac{1}{2} \sum_{i=1}^N \alpha x_i^2 \partial_{x_i}^2 + \sum_{1 \leq i \neq j \leq N} \frac{x_i x_j}{x_i - x_j} \left(\partial_{x_i} - \frac{\theta_i - \theta_j}{x_i - x_j} \partial_{\theta_i} \right) \quad (3.1)$$

$$\Delta = \sum_{i=1}^N \alpha x_i \theta_i \partial_{x_i} \partial_{\theta_i} + \sum_{1 \leq i \neq j \leq N} \frac{x_i \theta_j + x_j \theta_i}{x_i - x_j} \partial_{\theta_i}. \quad (3.2)$$

When $\theta_i = 0$, Δ disappears and D reduces to the operator whose eigenfunctions are the ordinary Jack polynomials. The issue concerning the possible degeneracies in the eigenvalues of D and Δ is discussed below.

In order to fully define the sJacks, two more objects are required. We first need to introduce the symmetric supermonomials [16]:

$$m_\Lambda = \sum'_{\sigma \in S_N} \theta_{\sigma(1)} \cdots \theta_{\sigma(m)} x_{\sigma(1)}^{\Lambda_1} \cdots x_{\sigma(N)}^{\Lambda_N}, \quad (3.3)$$

where the prime indicates a sum over distinct permutations of S_N . For instance, for $N = 3$,

$$m_{(2;1,1)} = \theta_1 x_1^2 x_2 x_3 + \theta_2 x_2^2 x_1 x_3 + \theta_3 x_3^2 x_1 x_2, \quad (3.4)$$

$$m_{(2,1;1)} = \theta_1 \theta_2 (x_1^2 x_2 - x_2^2 x_1) x_3 + \theta_1 \theta_3 (x_1^2 x_3 - x_3^2 x_1) x_2 + \theta_2 \theta_3 (x_2^2 x_3 - x_3^2 x_2) x_1. \quad (3.5)$$

More examples are displayed in Tables 1 and 3 of [16].

We also need a partial order that allows to compare superpartitions and that generalizes the usual dominance order $<$ on partitions. For technical reasons, we postpone to Section 3.3 the definition of the generalized dominance order, still denoted $<$. What is important for the moment is the following: the supermonomial m_Ω is said to be lower than m_Λ whenever Ω and Λ have the same degree $(n|m)$ and $\Omega < \Lambda$.

We are now in position to provide a practical characterization of the sJacks. For each superpartition Λ , the sJack $P_\Lambda^{(\alpha)}$ is the unique symmetric superpolynomial satisfying the unitriangularity condition

$$P_\Lambda^{(\alpha)} = m_\Lambda + \text{lower terms}, \quad (3.6)$$

and the double-eigenfunction condition

$$D P_\Lambda^{(\alpha)} = \varepsilon_\Lambda(\alpha) P_\Lambda^{(\alpha)} \quad \text{and} \quad \Delta P_\Lambda^{(\alpha)} = \epsilon_\Lambda(\alpha) P_\Lambda^{(\alpha)}, \quad (3.7)$$

where $\varepsilon_\Lambda(\alpha)$ and $\epsilon_\Lambda(\alpha)$ denote the eigenvalues. As shown in [17], the above definition together with the exact formula for the action of D and Δ on the monomial basis directly lead to explicit bi-determinantal formulas for the expansion

$$P_\Lambda^{(\alpha)} = \sum_{\Omega} c_{\Lambda\Omega} m_\Omega. \quad (3.8)$$

For a given degree $(n|m)$, the coefficient $c_{\Lambda\Omega}$ can be viewed as the entry labeled by Λ, Ω in the transition matrix from the sJack basis to the supermonomial basis. The simplest transitions matrices are given in Appendix C.

It is worth noting that the eigenvalues of the operators D and Δ are degenerate. Indeed, for sufficiently high degrees, one can find pairs of superpartitions $\Lambda \neq \Omega$ such that both $\varepsilon_\Lambda(\alpha) = \varepsilon_\Omega(\alpha)$ and $\epsilon_\Lambda(\alpha) = \epsilon_\Omega(\alpha)$. One might be tempted to conclude from this that the above characterization of the sJacks is not unique. But it is nothing of the sort. The crucial point is that the unitriangularity condition (3.6) suffices to guarantee the uniqueness. This follows from the fact that when two superpartitions $\Lambda \neq \Omega$ are comparable in the dominance order (i.e., either $\Lambda < \Omega$ or $\Lambda > \Omega$), then necessarily $(\varepsilon_\Lambda(\alpha), \epsilon_\Lambda(\alpha)) \neq (\varepsilon_\Omega(\alpha), \epsilon_\Omega(\alpha))$ [17].

A further remark is in order. Implicitly, the above definition for the sJacks depends on the number N of variables (x_i, θ_i) . However, it can be shown that the expansion (3.8) in the monomial basis is stable with respect N : suppose that N is greater than the bosonic degree of the superpartition Λ , then as N increases, the coefficients $c_{\Lambda\Omega}$ in (3.8) remain the same. In other words, the sJacks remain

well-defined in the situation where the number of variables is infinite (in which case they are called functions rather than polynomials but this distinction is irrelevant for our purposes). But an infinite number of variables is precisely the situation at hand, the reason being essentially rooted in that there are infinitely different modes in the Fock spaces.

3.2 Relation with the power-sums

As we have seen earlier, the correspondence between states in the Fock modules and symmetric superpolynomials is more easily understood in terms of power-sums than in terms of monomials. We thus need to rewrite the sJacks as polynomials in the power-sums.

We recall that each element in a basis of the space \mathcal{R} of symmetric superpolynomials in infinitely many variables is indexed by a superpartition. Both the sJacks and the products of power-sums, defined in (2.25), provide bases of \mathcal{R} . Consequently, we can write

$$P_{\Lambda}^{(\alpha)} = \sum_{\Omega} e_{\Lambda\Omega} p_{\Omega} \quad \text{and} \quad p_{\Lambda} = \sum_{\Omega} \tilde{e}_{\Lambda\Omega} P_{\Omega}^{(\alpha)}, \quad (3.9)$$

for some expansion coefficients $e_{\Lambda\Omega}$ and $\tilde{e}_{\Lambda\Omega}$. These coefficients can be explicitly calculated using the known expansion (3.8) and the simple transformation rules relating the p_{Ω} and the m_{Λ} 's. Appendix C contains all the coefficients $e_{\Lambda\Omega}$ for the superpartitions of bosonic degree less than 4.

From the second relation in (3.9) and the \mathcal{F} - \mathcal{R} correspondence given in (2.28), we see that any state in a NS highest-weight module can be represented as a linear combination of sJacks:

$$\sum_{\text{level}(\Lambda)=k} c_{\Lambda} d_{-\Lambda} |\eta\rangle \longleftrightarrow \sum_{\text{level}(\Lambda)=k} b_{\Lambda}(\alpha) P_{\Lambda}^{(\alpha)}, \quad (3.10)$$

for some constants $b_{\Lambda}(\alpha)$.

The power-sums-basis provides an alternative definition of the sJacks that avoids reference to the differential operators D and Δ , which depend upon the number N of variables. For this, we need the natural scalar product on \mathcal{R} , which is defined in terms of the power-sums as

$$\langle\langle p_{\Lambda} | p_{\Omega} \rangle\rangle_{\alpha} = (-1)^{\binom{m}{2}} \alpha^{\ell(\Lambda)} z_{\Lambda^s} \delta_{\Lambda,\Omega}, \quad (3.11)$$

where z_{Λ^s} is given by

$$z_{\Lambda^s} = \prod_{i \geq 1} i^{n_{\Lambda^s}(i)} n_{\Lambda^s}(i)!, \quad (3.12)$$

with $n_{\Lambda^s}(i)$ being the number of parts in Λ^s equal to i . It turns out that the sJacks are orthogonal with respect to the scalar product (3.11):

$$\langle\langle P_{\Lambda}^{(\alpha)} | P_{\Omega}^{(\alpha)} \rangle\rangle_{\alpha} = 0 \quad \text{when} \quad \Lambda \neq \Omega \quad (3.13)$$

Actually, to define the sJacks, this orthogonality condition can be used instead of the double-eigenfunction characterization presented above. To be clear, the sJacks can be defined uniquely from the two conditions (3.6) and (3.13). Note that the norm of $P_{\Lambda}^{(\alpha)}$ with respect to the above scalar product is computable and will be given below.

3.3 Diagrams for superpartitions

A key step in our finding of the general formula for the sJack representation of the singular vectors was a precise diagrammatic characterization of the contributing sJacks at a given level (this is the subject of Sect. 5.1). This relies on the diagram representation of the superpartitions, which we review here.

$$\Lambda = (3, 1, 0; 2, 1) = (\textcircled{3}, 2, \textcircled{1}, 1, \textcircled{0}) : \begin{array}{c} \square \square \square \bigcirc \\ \square \square \square \\ \square \bigcirc \\ \square \\ \bigcirc \end{array} . \quad (3.14)$$
$$\Lambda^{\oplus}: \begin{array}{|c|} \hline \\ \hline \end{array}, \quad \Lambda^*: \begin{array}{|c|} \hline \\ \hline \end{array}. \tag{3.15}$$
$$\text{level}(\Lambda) = \frac{1}{2}(|\Lambda^*| + |\Lambda^{\otimes}|). \quad (3.16)$$
$$\Omega \leq \Lambda \quad \text{iff} \quad \Omega^* \leq \Lambda^* \quad \text{and} \quad \Omega^{**} \leq \Lambda^{**},$$
$$\lambda \geq \mu \quad \Longleftrightarrow \quad |\lambda| = |\mu| \quad \text{and} \quad \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \quad \text{for all } i. \quad (3.17)$$
$$\langle\langle P_{\Lambda}^{(\alpha)} | P_{\Lambda}^{(\alpha)} \rangle\rangle_{\alpha} = (-1)^{\binom{m}{2}} j_{\Lambda}, \quad (3.18)$$
$$j_{\Lambda}(\alpha) = \alpha^m \prod_{s \in \Lambda} \frac{h_{\Lambda}^{\uparrow}(s)}{h_{\Lambda}^{\downarrow}(s)}, \quad (3.19)$$
$$\begin{aligned} h_{\Lambda}^{\uparrow}(s) &= l_{\Lambda^{\oplus}}(s) + \alpha(a_{\Lambda^*}(s) + 1) \\ h_{\Lambda}^{\downarrow}(s) &= l_{\Lambda^*}(s) + 1 + \alpha a_{\Lambda^{\oplus}}(s). \end{aligned} \quad (3.20)$$
$$a_\lambda(s) = \lambda_i - j \quad \text{and} \quad l_\lambda(s) = \lambda'_j - i, \quad (3.21)$$

where λ' stands for the conjugate of λ , obtained by interchanging rows and columns.

The expression for the norm will enter in the construction of the dual of a singular vector. Furthermore, a particular ratio of upper and lower hooks will be a basic building block of the general formula for the singular vectors.

4 Superconformal singular vectors as sJacks: simplest cases

4.1 Strategy

Let us denote by \mathcal{G}_k , as we did in Section 2.4, the representation of the operators G_k obtained by replacing in (2.8) the modes a_n and b_l by the corresponding power-sums or their derivatives via (2.19). The operators \mathcal{G}_k depend upon the parameters α , η , and γ .

Now, considering that we are interested in singular vectors of Kac modules, whose central charge $c(t)$ and conformal dimension $h_{r,s}(t)$ are given by (2.6), we have to connect the latter parameters to γ , η and α . We recall that in the representation over the space of symmetric superpolynomials, γ determines the central charge via $c = 3/2 - 12\gamma^2$, while for the Kac modules, $c(t) = 15/2 - 3(t + t^{-1})$. We thus have $\gamma = \pm(t - 1)/(2\sqrt{t})$. For convenience, we set

$$\gamma = \frac{(t - 1)}{2\sqrt{t}}. \quad (4.1)$$

Moreover, we know that $h = \eta^2/2 - \gamma\eta$. Once again, two solutions are possible, but only one complies with $\gamma = [\eta]_{r=s=0}$. We then choose

$$\eta = \frac{t}{8}(r^2 - 1) + \frac{1}{8t}(s^2 - 1) - \frac{1}{4}(rs - 1). \quad (4.2)$$

It remains to relate α and t . For this, we first remark that the change $t \mapsto t^{-1}$ induces a duality between the Kac module labeled by r, s to that labeled by s, r :

$$c(t) = c(t^{-1}) \quad h_{r,s}(t) = h_{s,r}(t^{-1}). \quad (4.3)$$

As will become clear in Section 4.6, the sJacks also satisfy a duality property, but now based on the transformation $\alpha \mapsto \alpha^{-1}$. (For the perspective of the supersymmetric Calogeo-Sutherland model, this is not surprising since the model possesses a strong-weak coupling constant symmetry.) In order to simultaneously make use of these two dualities, we must set

$$\alpha \equiv t. \quad (4.4)$$

As a consequence, from now on, we assume that each Kac module is fully determined by the values of r , s , and α .

Since the singular vector $|\chi_{r,s}\rangle^{\text{NS}}$ is a descendant that behaves as a highest-weight state, to identify the representative of $|\chi_{r,s}\rangle^{\text{NS}}$ in the space \mathcal{R} of symmetric superpolynomials, it suffices to verify the differential-operator version of the conditions (2.4) on a non-trivial element $f \in \mathcal{R}$. We thus search linear combinations $\sum v_\Lambda P_\Lambda^{(\alpha)}$ of fixed level $rs/2 = n + m/2$ (recall that $n = |\Lambda|$ and m is the fermionic degree of Λ) satisfying

$$\mathcal{G}_{\frac{1}{2}}\left(\sum v_\Lambda P_\Lambda^{(\alpha)}\right) = 0 \quad \text{and} \quad \mathcal{G}_{\frac{3}{2}}\left(\sum v_\Lambda P_\Lambda^{(\alpha)}\right) = 0 \quad (4.5)$$

when

$$\eta_{r,s} = \frac{1}{2\sqrt{\alpha}}\left((1+r)\alpha - (1+s)\right), \quad \text{and} \quad \gamma = \frac{(\alpha - 1)}{2\sqrt{\alpha}}. \quad (4.6)$$

Note that since $r + s$ is even in the NS sector, r and s are either both even or both odd. In the former case, the level is integer, so that m is necessarily even (the singular vector is bosonic), while in the latter case, m is odd (and the singular vector is fermionic). We stress that the precise normalization of the solutions of the system (4.5) is irrelevant.

The explicit expressions of $\mathcal{G}_{\frac{1}{2}}$ and $\mathcal{G}_{\frac{3}{2}}$ are given in (2.31) and (2.34) respectively. To calculate the action of these operators on a given sJack $P_{\Lambda}^{(\alpha)}$, one first needs to expand $P_{\Lambda}^{(\alpha)}$ in the power-sum basis. For example, given that $P_{(0;)}^{(\alpha)}$ is equal to $p_{(0;)} = \tilde{p}_0$, we get

$$\mathcal{G}_{\frac{1}{2}} P_{(0;)}^{(\alpha)} = (\bar{\eta} + 1 - \alpha) \quad \text{and} \quad \mathcal{G}_{\frac{3}{2}} P_{(0;)}^{(\alpha)} = 0. \quad (4.7)$$

From this simple calculation, we identify a first singular-vector representation: $P_{(0;)}^{(\alpha)}$ when $\bar{\eta} = \alpha - 1$, that is, $\eta = \eta_{1,1}$. In other words, we have

$$|\chi_{1,1}\rangle^{\text{NS}} = G_{-\frac{1}{2}}|0\rangle^{\text{NS}} \longleftrightarrow P_{(0;)}^{(\alpha)}. \quad (4.8)$$

The next simplest example is

$$P_{(0;1)}^{(\alpha)} = p_{(0;1)} - p_{(1;0)} = \tilde{p}_0 p_1 - \tilde{p}_1, \quad (4.9)$$

for which

$$\mathcal{G}_{\frac{1}{2}}(\tilde{p}_0 p_1 - \tilde{p}_1) = (\bar{\eta} + 2 - \alpha) p_{(1)} \quad (4.10)$$

$$\mathcal{G}_{\frac{3}{2}}(\tilde{p}_0 p_1 - \tilde{p}_1) = (\bar{\eta} + 2 - \alpha). \quad (4.11)$$

We see that the terms on the rhs vanish if $\eta = \eta_{1,3}$. We thus obtain a second singular vector representation:

$$|\chi_{1,3}\rangle^{\text{NS}} \longleftrightarrow P_{(0;1)}^{(\alpha)}. \quad (4.12)$$

Proceeding similarly and using the fact that

$$P_{(1;)}^{(\alpha)} = \frac{1}{(1 + \alpha)}(p_{(0;1)} + \alpha p_{(1;)}), \quad (4.13)$$

we find

$$|\chi_{3,1}\rangle^{\text{NS}} \longleftrightarrow P_{(1;)}^{(\alpha)}. \quad (4.14)$$

4.2 Sample NS singular vectors in the sJack basis

The three examples just worked out are too simple to provide any clue concerning the general pattern of the sJack-representation of the singular vectors at higher levels. At level 2 for example, there is only one possible singular vector, which can be found in the Kac module with $r = s = 2$. Simple manipulations lead to

$$|\chi_{2,2}\rangle^{\text{NS}} \longleftrightarrow P_{(2)}^{(\alpha)} + \frac{2}{(1 + \alpha)} P_{(1,1)}^{(\alpha)} + \frac{4}{(1 + \alpha)} P_{(1,0;)}^{(\alpha)}. \quad (4.15)$$

At first sight, the last representation seems disappointing since it contains all possible superpartitions whose level $n + m/2$ is equal to 2. However, as we will see shortly, level 2 is the only case where the sJack basis does not show any particular advantage. Indeed, at level $5/2$, we get two very simple representations:

$$|\chi_{1,5}\rangle^{\text{NS}} \longleftrightarrow P_{(0;1,1)}^{(\alpha)} \quad \text{and} \quad |\chi_{5,1}\rangle^{\text{NS}} \longleftrightarrow P_{(2;)}^{(\alpha)}. \quad (4.16)$$

At level 3, there is no singular vector, but level $7/2$ exhibits once again a remarkable simplicity:

$$|\chi_{1,7}\rangle^{\text{NS}} \longleftrightarrow P_{(0;1,1,1)}^{(\alpha)} \quad \text{and} \quad |\chi_{7,1}\rangle^{\text{NS}} \longleftrightarrow P_{(3;)}^{(\alpha)} \quad (4.17)$$

At level 4, there are two singular vectors:

$$\begin{aligned} |\chi_{2,4}\rangle^{\text{NS}} \longleftrightarrow & P_{(2,2)}^{(\alpha)} + \frac{2}{(\alpha+1)} P_{(2,1,1)}^{(\alpha)} + \frac{12}{(\alpha+3)(\alpha+2)} P_{(1,1,1,1)}^{(\alpha)} \\ & + \frac{4}{(\alpha+1)} P_{(1,0;2)}^{(\alpha)} + \frac{8}{(\alpha+3)(\alpha+2)} P_{(1,0;1,1)}^{(\alpha)} \end{aligned} \quad (4.18)$$

$$\begin{aligned} |\chi_{4,2}\rangle^{\text{NS}} \longleftrightarrow & P_{(4)}^{(\alpha)} + \frac{4}{(3\alpha+1)} P_{(3,1)}^{(\alpha)} + \frac{2(\alpha+2)}{(2\alpha+1)(\alpha+1)} P_{(2,2)}^{(\alpha)} \\ & + \frac{8}{(3\alpha+1)} P_{(3,0;)}^{(\alpha)} + \frac{16}{(2\alpha+1)} P_{(2,1;)}^{(\alpha)}. \end{aligned} \quad (4.19)$$

These representations in the sJack basis contain 5 terms, but in the power-sum basis (or in any other “classical” basis), they would contain exactly 10 terms, which corresponds to the number of superpartitions at level 4 (and the number of independent states in the NS Verma module). What we observed by comparing the sJack-representations at levels 2 and 4 is an insight of a more general phenomenon for $r, s > 1$: the greater is the level $rs/2$ of $|\chi_{r,s}\rangle^{\text{NS}}$, the greater is the difference between the number of terms in the sJack-representation of the singular vector and the total number of superpartitions at level $rs/2$.

4.3 NS singular vectors with $r = 1$ or $s = 1$

The examples given in equations (4.8), (4.12), (4.14), (4.16), and (4.17) suggest that there are two infinite sequences of singular vectors that can be represented by a single sJack. Indeed, one can show that

$$\mathcal{G}_{\frac{1}{2}} P_{(0;1^k)}^{(\alpha)} = (\bar{\eta} + k + 1 - \alpha) P_{(1^k)}^{(\alpha)} \quad \text{and} \quad \mathcal{G}_{\frac{3}{2}} P_{(0;1^k)}^{(\alpha)} = (\bar{\eta} + k + 1 - \alpha) P_{(1^{k-1})}^{(\alpha)}, \quad (4.20)$$

where 1^k denotes the partition whose k parts are equal to 1. This implies that $P_{(0;1^k)}^{(\alpha)}$ can be associated to a singular vector whenever $\bar{\eta} = \alpha - k - 1$, which is equivalent to $\eta = \eta_{1,s}$ with $s = 2k + 1$. For $r = 1$ and s odd, we thus have

$$\boxed{|\chi_{1,s}\rangle^{\text{NS}} \longleftrightarrow P_{(0;1^{\frac{s-1}{2}})}^{(\alpha)}} \quad (4.21)$$

The dual version of the above singular vector is, with r odd,

$$\boxed{|\chi_{r,1}\rangle^{\text{NS}} \longleftrightarrow P_{(\frac{r-1}{2};)}^{(\alpha)}} \quad (4.22)$$

This identification follows from

$$\mathcal{G}_{\frac{1}{2}} P_{(k;)}^{(\alpha)} = \frac{(\bar{\eta} + 1 - (k+1)\alpha)}{(1+k\alpha)} P_{(k)}^{(\alpha)} \quad \text{and} \quad \mathcal{G}_{\frac{3}{2}} P_{(k;)}^{(\alpha)} = \frac{k\alpha(\bar{\eta} + 1 - (k+1)\alpha)}{(1+(k-1)\alpha)(1+k\alpha)} P_{(k-1)}^{(\alpha)}. \quad (4.23)$$

The singular vectors (4.21) and (4.22) are the only NS ones we found for which there is a representation in terms of a single $P_{\Lambda}^{(\alpha)}$ for a generic value of α . In both cases, $m = 1$, so that the levels are respectively $s/2$ and $r/2$, as they should.

4.4 NS singular vectors for $r = 2$

Equations (4.15) and (4.18) provide the simplest examples of singular vectors with $r = 2$. The next singular vector of this type appears at level 6:

$$\begin{aligned} |\chi_{2,6}\rangle^{\text{NS}} \longleftrightarrow & P_{(2,2,2)}^{(\alpha)} + \frac{2}{(\alpha+1)} P_{(2,2,1,1)}^{(\alpha)} \\ & + \frac{12}{(\alpha+3)(\alpha+2)} P_{(2,1,1,1,1)}^{(\alpha)} + \frac{4}{(\alpha+1)} P_{(1,0;2,2)}^{(\alpha)} + \frac{8}{(\alpha+3)(\alpha+2)} P_{(1,0;2,1,1)}^{(\alpha)} \\ & + \frac{120}{(\alpha+3)(\alpha+4)(\alpha+5)} P_{(1,1,1,1,1,1)}^{(\alpha)} + \frac{48}{(\alpha+3)(\alpha+4)(\alpha+5)} P_{(1,0;1,1,1,1)}^{(\alpha)} \end{aligned} \quad (4.24)$$

Note that the coefficient $P_{(2^{s/2})}^{(\alpha)}$ is set to 1, a normalization compatible with that of (4.15) and (4.18). By inspection, we obtained the following general expression:

$$|\chi_{2,s}\rangle^{\text{NS}} \longleftrightarrow \sum_{\ell=0}^{s/2} \frac{\prod_{i=1}^{\ell} (4i-2)}{\prod_{i=\ell}^{2\ell-1} (\alpha+i)} P_{(2^{s/2-\ell} 1^{2\ell})}^{(\alpha)} + \sum_{\ell'=1}^{s/2} \frac{4 \prod_{i=1}^{\ell'-1} (4i-2)}{\prod_{i=\ell'}^{2\ell'-1} (\alpha+i)} P_{(1,0;2^{s/2-\ell'} 1^{2\ell'-1})}^{(\alpha)} \quad (4.25)$$

with the understanding that if the upper bound is smaller than the lower one in the products, then the product is replaced by 1. This formula has been tested up to level 16, that is, for all s even and ≤ 16 .

4.5 NS singular vectors for $r = 3$

Up to this point, only one singular vector with $r = 3$ has been given, namely $|\chi_{3,1}\rangle^{\text{NS}}$ in (4.14). Let us display a few more examples conveniently normalized:

$$|\chi_{3,3}\rangle^{\text{NS}} \longleftrightarrow P_{(1;3)}^{(\alpha)} + \frac{(\alpha+2)(3\alpha+1)}{2(\alpha+1)^3} P_{(1;2,1)}^{(\alpha)} - \frac{(3\alpha+1)}{(\alpha+1)^2} P_{(2,1;0;0)}^{(\alpha)} \quad (4.26)$$

$$\begin{aligned} |\chi_{3,5}\rangle^{\text{NS}} \longleftrightarrow & P_{(1;3,3)}^{(\alpha)} + \frac{(\alpha+2)(3\alpha+1)}{2(\alpha+1)^3} P_{(1;3,2,1)}^{(\alpha)} + \frac{3(\alpha+4)(3\alpha+1)}{(\alpha+2)^2(2\alpha+3)(\alpha+1)} P_{(1;2,2,1,1)}^{(\alpha)} \\ & - \frac{(3\alpha+1)}{(\alpha+1)^2} P_{(2,1,0;3)}^{(\alpha)} - \frac{3(3\alpha+1)}{(2\alpha+3)(\alpha+2)(\alpha+1)} P_{(2,1,0;2,1)}^{(\alpha)} \end{aligned} \quad (4.27)$$

$$\begin{aligned} |\chi_{3,7}\rangle^{\text{NS}} \longleftrightarrow & P_{(1;3,3,3)}^{(\alpha)} + \frac{(\alpha+2)(3\alpha+1)}{2(\alpha+1)^3} P_{(1;3,3,2,1)}^{(\alpha)} + \frac{3(\alpha+4)(3\alpha+1)}{(\alpha+2)^2(2\alpha+3)(\alpha+1)} P_{(1;3,2,2,1,1)}^{(\alpha)} \\ & + \frac{9(\alpha+6)(3\alpha+1)(\alpha+4)(3\alpha+5)}{2(3+\alpha)^3(2\alpha+5)(\alpha+1)(\alpha+2)^2} P_{(1;2,2,2,1,1,1)}^{(\alpha)} - \frac{(3\alpha+1)}{(\alpha+1)^2} P_{(2,1,0;3,3)}^{(\alpha)} \\ & - \frac{3(3\alpha+1)}{(2\alpha+3)(\alpha+2)(\alpha+1)} P_{(2,1,0;3,2,1)}^{(\alpha)} - \frac{3(\alpha+4)(3\alpha+1)(3\alpha+5)}{(2\alpha+5)(\alpha+1)(3+\alpha)^2(\alpha+2)^2} P_{(2,1,0;2,2,1,1)}^{(\alpha)} \end{aligned} \quad (4.28)$$

By inspection of these cases, we were led to the following formula.

$$\begin{aligned} |\chi_{3,s}\rangle^{\text{NS}} \longleftrightarrow & \sum_{\ell=0}^{(s-1)/2} \prod_{i=1}^{\ell} \frac{i(3\alpha+2i-1)(\alpha+2i)}{(\alpha+i)^2(2\alpha+\ell+i)} P_{(1;3^{(s-1)/2-\ell} 2^{\ell} 1^{\ell})}^{(\alpha)} \\ & - \frac{(3\alpha+1)}{(\alpha+1)^2} \sum_{\ell'=0}^{(s-3)/2} \prod_{i=1}^{\ell'} \frac{i(3\alpha+2i+1)(\alpha+2i)}{(\alpha+i+1)^2(2\alpha+\ell'+i+1)} P_{(2,1,0;3^{(s-3)/2-\ell'} 2^{\ell'} 1^{\ell'})}^{(\alpha)} \end{aligned}$$

where again, we set $\prod_{i=1}^0 f(i) = 1$. This explicit formula has been tested up to level $33/2$, that is, for all s odd and ≤ 11 .

4.6 Relating the singular vector $|\chi_{r,s}\rangle$ to $|\chi_{s,r}\rangle$

In the previous two subsections, we have presented two explicit formulas for which $r \leq s$. We now show how to interchange the two labels, which would thereby amount to recover the cases $r \geq s$ from those with the inequality reversed.

Interchanging the labels r and s in the expression for $h_{r,s}$ can be obtained by interchanging $t \rightarrow 1/t$ – see eq. (2.6). In the representation of the singular vectors, this appears to be slightly more complicated. Indeed, neither $|\chi_{2,2}\rangle^{\text{NS}}$ in (4.15) nor $|\chi_{3,3}\rangle^{\text{NS}}$ in (4.26) are invariant under $\alpha \leftrightarrow 1/\alpha$. Similarly, $|\chi_{2,4}\rangle^{\text{NS}}$ and $|\chi_{4,2}\rangle^{\text{NS}}$ are not related by such a simple transformation (cf. the expressions (4.18) and (4.19)).

The correct rule for transforming $|\chi_{r,s}\rangle$ to $|\chi_{s,r}\rangle$ involves the duality transformation of the $P_\Lambda^{(\alpha)}$'s (see [36, 43] for the ordinary Jack polynomials and [19] in the super-case). Up to an irrelevant multiplicative factor, the rule is

$$\text{if } |\chi_{r,s}\rangle^{\text{NS}} \longleftrightarrow \sum_{\Lambda} v_{\Lambda}(\alpha) P_{\Lambda}^{(\alpha)} \quad \text{then} \quad |\chi_{s,r}\rangle^{\text{NS}} \longleftrightarrow \left[\sum_{\Lambda} v_{\Lambda}(\alpha) \tilde{\omega}_{\alpha} P_{\Lambda}^{(\alpha)} \right]_{\alpha \rightarrow 1/\alpha}, \quad (4.29)$$

where⁴

$$\tilde{\omega}_{\alpha} P_{\Lambda}^{(\alpha)} = j_{\Lambda'}(\alpha) P_{\Lambda'}^{(1/\alpha)}. \quad (4.31)$$

In the last equation, Λ' stands for the conjugate of Λ : its diagram is obtained from that of Λ by interchanging rows and columns. Finally, $j_{\Lambda}(\alpha)$ denotes the norm squared of $P_{\Lambda}^{(\alpha)}$ defined in eqs (3.18) and (3.19).

Let us consider as an example, the construction of $|\chi_{4,2}\rangle^{\text{NS}}$ out of $|\chi_{2,4}\rangle^{\text{NS}}$ whose expression is given in (4.18). The transformation of $|\chi_{2,4}\rangle^{\text{NS}}$ according to the rule (4.29) yields:

$$\begin{aligned} & \bar{j}_{(2,2)} P_{(2,2)'}^{(\alpha)} + \bar{j}_{(2,1,1)} \frac{2\alpha}{(\alpha+1)} P_{(2,1,1)'}^{(\alpha)} + \bar{j}_{(1,1,1,1)} \frac{12\alpha^2}{(3\alpha+1)(2\alpha+1)} P_{(1,1,1,1)'}^{(\alpha)} \\ & + \bar{j}_{(1,0;2)} \frac{4\alpha}{(\alpha+1)} P_{(1,0;2)'}^{(\alpha)} + \bar{j}_{(1,0;1,1)} \frac{8\alpha^2}{(3\alpha+1)(2\alpha+1)} P_{(1,0;1,1)'}^{(\alpha)} \end{aligned} \quad (4.32)$$

where \bar{j}_{Λ} stands for $j_{\Lambda}(1/\alpha)$ and the different $j_{\Lambda}(\alpha)$ are

$$\begin{aligned} j_{(2,2)} &= \frac{(2\alpha+1)\alpha^2}{\alpha+2}, & j_{(2,1,1)} &= \frac{(\alpha+1)^2\alpha^2}{\alpha+3}, & j_{(1^4)} &= \frac{1}{24}(\alpha+3)(\alpha+2)(\alpha+1)\alpha \\ j_{(1,0;2)} &= \frac{2(\alpha+1)^2\alpha^2}{\alpha+2}, & j_{(1,0;1,1)} &= \frac{1}{2}(\alpha+1)(\alpha+2)\alpha^2. \end{aligned} \quad (4.33)$$

Substituting these expressions in the previous equation yields $[(\alpha+1)/2\alpha^2] |\chi_{4,2}\rangle$, where the representation of $|\chi_{4,2}\rangle^{\text{NS}}$ is given in (4.19).

⁴The operator $\tilde{\omega}_{\alpha}$ is given by $\tilde{\omega}_{\alpha} = (-1)^{\binom{m}{2}} \hat{\omega}_{\alpha}$, where the later is defined in [19] – cf. Sect. 6.1 there. The action of $\hat{\omega}_{\alpha}$ on $P_{\Lambda}^{(\alpha)}$ is not the natural way of defining this duality operation. It is actually defined as a mild transformation on the free-field representatives, that is, on the power sums:

$$\hat{\omega}_{\alpha}(p_n) = (-1)^{n-1} \alpha p_n \quad \text{and} \quad \hat{\omega}_{\alpha}(\tilde{p}_n) = (-1)^n \alpha \tilde{p}_n. \quad (4.30)$$

It is somewhat amazing that, when lifted at the level of sJacks, it yields a transformation as simple as (4.31).

5 Superconformal singular vectors as sJacks: general formula

In the previous section, we presented certain closed-form representations $\sum v_\Lambda P_\Lambda^{(\alpha)}$ of the NS singular vectors. A first observation is that all the coefficients v_Λ factorize nicely into products and ratios of terms $(i\alpha + j)$ for nonnegative integers i, j . This suggests an underlying natural combinatorial description. A second observation is that not all superpartitions of a given degree $rs/2$ do occur in the sum $\sum v_\Lambda P_\Lambda^{(\alpha)}$. For instance, in the representation of $|\chi_{1,5}\rangle$, only $(0; 1, 1)$ contributes, so that the following superpartitions of level $5/2$ are missing: $(2; 0)$, $(1; 1)$, $(0; 2)$.

In the following, we give a precise characterization of the sJacks at level $rs/2$ that appear in the representation of the (r, s) singular vector, and then present an explicit conjectural expression for the coefficients v_Λ .

5.1 Self-complementary superpartitions

A severe restriction selecting the partitions labeling the polynomials representing the singular vectors was already observed in the non-supersymmetric case: only the partitions of degree rs whose diagram fills perfectly the rectangular shape with r columns and s rows do contribute. Of course, this criterion singles out the partition (r^s) .

It turns out that a similar but less restrictive criterion selects the superpartitions that occur in the representation of the NS singular vector at level $rs/2$. Recall that a superpartition Λ is equivalent to a pair of partitions Λ^* and Λ° (cf. Sect. 3.3). Now, the superpartitions appearing in the representation of $|\chi_{r,s}\rangle^{\text{NS}}$ are level- $rs/2$ superpartitions for which the complement of the diagram of Λ° in the rectangle with r columns and s rows corresponds to the diagram of Λ^* rotated by π . The set of such superpartitions will be denoted $\square_{r,s}$, and a superpartition belonging to $\square_{r,s}$ will be called (r, s) -self-complementary. Take for instance the level- $15/2$ superpartition $(2, 1, 0; 2, 1)$; the $(3, 5)$ -self-complementarity is illustrated in the following diagram, where the diagram with thick frames correspond to that of Λ^* rotated by π and where we have used the diagram of Λ instead of that of Λ° (whose outer shapes are the same) to better illustrate the self-complementarity:

(5.1)

$(2, 1, 0; 2, 1)$ is thus an element of $\square_{3,5}$. The complete set of $(3, 5)$ -self-complementary superpartitions is

$$\square_{3,5} = \{(1; 3, 3), (1; 3, 2; 1), (1; 2^2, 1^2), (2, 1, 0; 3), (2, 1, 0; 2, 1)\}. \quad (5.2)$$

and their corresponding diagrams are

(5.3)

Similarly, the diagrams corresponding to the $(2, 6)$ -self-complementary superpartitions:

$$\square_{2,6} = \{(2^3), (2^2, 1^2), (2, 1^4), (1^6), (1, 0; 2^2), (1, 0; 2, 1^2), (1, 0; 1^4)\}, \quad (5.4)$$

are:

(5.5)

That only Jack superpolynomials whose superpartition belongs to $\square_{r,s}$ appear in the representation of $|\chi_{r,s}\rangle$ has been verified for all our examples (which reach level $25/2$). This rule explains the non-occurrence of the level- $5/2$ superpartitions $(2; 0)$, $(1; 1)$, $(0; 2)$ in the representation of $|\chi_{1,5}\rangle$ since they are not $(1, 5)$ -self-complementary. Actually, $\square_{1,5}$, and more generally $\square_{1,s}$, contains a single element and the same is true for $\square_{r,1}$.

Finally, it should be clear that $\Lambda \in \square_{r,s}$ implies that $\Lambda' \in \square_{s,r}$, a property that was implicitly assumed in the duality transformation of the previous subsection.

5.2 General formula for the coefficients of the sJack-form of the NS singular vectors

In this section, we present a conjecture for the explicit form of the coefficient v_Λ in the representation of

$$|\chi_{r,s}\rangle^{\text{NS}} \longleftrightarrow \sum_{\Lambda \in \square_{r,s}} v_\Lambda P_\Lambda^{(\alpha)}. \quad (5.6)$$

We stress that $\Lambda \in \square_{r,s}$ automatically implies that $\text{level}(\Lambda) = rs/2$. Let us fix at once our choice of normalization. For r and s even, the coefficient of the sJack labeled by $r^{s/2}$ is set equal to 1. Similarly, for r and s odd, the coefficient of the term with $\Lambda = ((r-1)/2; r^{(s-1)/2})$ is fixed to be 1. All our previous examples are compatible with this choice.

Note that, although the normalization of a singular vector is arbitrary, it turns out that by choosing the normalization appropriately was a key step in our finding of the general expression of the relative coefficients, because this particular choice reveals a clear recursive pattern. This pattern is manifest in the formulas for $r = 2$ and $r = 3$: changing $s \rightarrow s + 2$ produces extra terms but the coefficients of the terms that were present (e.g., terms with the same value of ℓ or ℓ' in (4.25)) are not affected.

Before displaying the general form of v_Λ , we first need to introduce some notation.

5.2.1 Introducing some combinatorial data We first identify two subsets of corners in a diagram [41]. The *removable corners* (resp. *addable corners*) of a diagram associated to a partition are the boxes that can be removed (resp. added) in such a way that the resulting diagram is still associated to a partition. The *interior corners* of a partition λ are the squares (i, j) such that the cell $(i+1, j+1)$ is an addable corner of λ . For instance, in $\lambda = (8, 4, 3, 3, 1, 1)$, we have marked the interior corners by an i and the removable corners by an r:

(5.7)

Next, we introduce a notion intrinsic to self-complementary superpartitions, namely that of *complementary circles*, together with some associated data. Let $\Lambda \in \square_{r,s}$. If Λ has a circle in row

$k \leq \lfloor s/2 \rfloor$, then it also has one in row $s+1-k$ by complementarity. Indeed, the presence of a circle in row k implies $\Lambda_k^\oplus - \Lambda_k^* = 1$, while the self-complementarity forces

$$\Lambda_k^\oplus + \Lambda_{s+1-k}^* = r \quad \text{and} \quad \Lambda_{s+1-k}^\oplus + \Lambda_k^* = r. \quad (5.8)$$

Hence $\Lambda_{s+1-k}^\oplus - \Lambda_{s+1-k}^* = \Lambda_k^\oplus - \Lambda_k^* = 1$, which means that there is a circle in row $s+1-k$. In this case, we say that the circles in rows k and $s+1-k$ are *complementary*, and denote the pair as (u, d) , where u and d are the coordinates of the cells corresponding to the circles in rows k and $s+1-k$ (u and d obviously stand respectively for up and down). When the number of circle is odd, there will remain an unpaired circle in row $(s+1)/2$ (s is then necessarily odd) which will play no role in the description of the general formula. Consider for example $\Lambda = (4, 2, 0; 4, 1) \in \square_{5,5}$:



We have only one pair (u, d) of complementary circles given by $u = (1, 5)$ and $d = (5, 1)$. Let us denote by $\Lambda_{(u,d)}$ the superpartition obtained by deleting the circle in position u and changing the circle in position d into a square. For the above example, $\Lambda_{(u,d)}$ is the diagram



We also let $\Lambda_{\text{rc}(u,d)}^\oplus$ (resp. $\Lambda_{\text{ic}(u,d)}^\oplus$) be the set of removable (resp. inner) corners of Λ^\oplus that lie strictly between cells u and d . For the diagram (5.9) the interior (i) and removable (r) corners of Λ^\oplus strictly between u and d are



Finally, given two cells $t_1 = (i_1, j_1)$ and $t_2 = (i_2, j_2)$ such that t_1 is southwest of t_2 , we let

$$\|t_1, t_2\| = i_1 - i_2 + \alpha(j_2 - j_1) \quad (5.12)$$

be the α -distance between t_1 and t_2 .

5.2.2 The explicit formula Let $(u_1, d_1), \dots, (u_\ell, d_\ell)$ be the pairs of complementary circles in Λ (which defines ℓ), and for convenience let $\lambda = \Lambda^\oplus$. Then, for $\Lambda \in \square_{r,s}$, the coefficient of $P_\Lambda^{(\alpha)}$ in the representation (5.6) of $|\chi_{r,s}\rangle^{\text{NS}}$, appears to be

$$v_\Lambda = 2^\ell (-1)^{s\ell} \prod_{j=1}^{\lfloor r/2 \rfloor} \left[f(r+2-2j, 2\lambda'_j - s) \prod_{i=s+1-\lambda'_j}^{\lambda'_j} \frac{\bar{h}_\Lambda^\downarrow(i, j)}{\bar{h}_\Lambda^\uparrow(i, j)} \right] \prod_{i=1}^\ell \left[\frac{\prod_{t \in \Lambda_{\text{rc}(u_i, d_i)}^\oplus} \|d_i, t\|}{\prod_{t \in \Lambda_{\text{ic}(u_i, d_i)}^\oplus} \|d_i, t\|} \right] \quad (5.13)$$

where $f(r, s)$ stands for the product:

$$f(r, s) = \begin{cases} \prod_{j=1}^{s/2} \frac{r\alpha + 2(j-1)}{(r-2)\alpha + 2j} & \text{if } r \text{ and } s \text{ are even,} \\ \frac{(r-1)\alpha}{(r-1)\alpha + 2} \prod_{j=1}^{(s-1)/2} \frac{r\alpha + 2j - 1}{(r-2)\alpha + 2j + 1} & \text{if } r \text{ and } s \text{ are odd.} \end{cases} \quad (5.14)$$

with $f(r, 0) = 1$ and where

$$\bar{h}_{\Lambda}^{\uparrow, \downarrow}(i, j) = \begin{cases} h_{\Lambda(u, d)}^{\uparrow, \downarrow}(i, j) & \text{if a certain } d \text{ lies in column } j \\ h_{\Lambda}^{\uparrow, \downarrow}(i, j) & \text{otherwise,} \end{cases} \quad (5.15)$$

where $h_{\Lambda}^{\uparrow, \downarrow}$ are defined in (3.20). Note that the product from $s+1-\lambda'_j$ to λ'_j is empty if $s+1-\lambda'_j > \lambda'_j$ (in which case it is replaced by 1).

We will illustrate the formula with the example $\Lambda = (4, 2, 0; 4, 1) \in \square_{5,5}$, whose diagram is displayed in (5.9). As already noticed, there is only one pair (u, d) of complementary circles ($u = (1, 5)$ and $d = (5, 1)$), so that $\ell = 1$; the first factor is thus $2(-1)^s = -2$. We have $\lambda' = (\Lambda^{\otimes})' = (5, 3, 3, 2, 1)$, and thus the f factors are

$$f(5, 5) = \frac{4\alpha}{(4\alpha + 2)} \frac{(5\alpha + 1)}{(3\alpha + 3)} \frac{(5\alpha + 3)}{(3\alpha + 5)} \quad \text{and} \quad f(3, 1) = \frac{2\alpha}{(2\alpha + 2)}. \quad (5.16)$$

When $j = 1$, since $\lambda'_1 = 5$ and $s+1-\lambda'_1 = 5+1-5 = 1$, the product of the ratio of hooks $h^{\downarrow}/h^{\uparrow}$ is over the boxes of the first column in $\lambda_{u,d}$ displayed in (5.10) (we use the diagram of $\Lambda_{u,d}$ rather than that of Λ because these boxes belong to the column of d). This factor is

$$\prod_{i=1}^5 \frac{\bar{h}^{\downarrow}(i, 1)}{\bar{h}^{\uparrow}(i, 1)} = \frac{(3\alpha + 5)}{(4\alpha + 4)} \frac{(3\alpha + 4)}{(4\alpha + 3)} \frac{(2\alpha + 3)}{(2\alpha + 2)} \frac{2}{(\alpha + 1)} \frac{1}{\alpha}. \quad (5.17)$$

When $j = 2$, since $\lambda'_2 = 3$ and $s+1-\lambda'_2 = 5+1-3 = 3$, the product is only on $i = 3$ and thus the hooks are those of cell $(3, 2)$ in Λ given by

$$\frac{\bar{h}^{\downarrow}(3, 2)}{\bar{h}^{\uparrow}(3, 2)} = \frac{(\alpha + 1)}{\alpha}. \quad (5.18)$$

Finally, the interior and removable corners of Λ^{\otimes} between u and d are indicated in (5.11). The product of their respective α -distances to d is

$$\frac{(2\alpha + 2)(3\alpha + 3)}{2(2\alpha + 3)(3\alpha + 4)}. \quad (5.19)$$

If we make the product of all these terms, we obtain

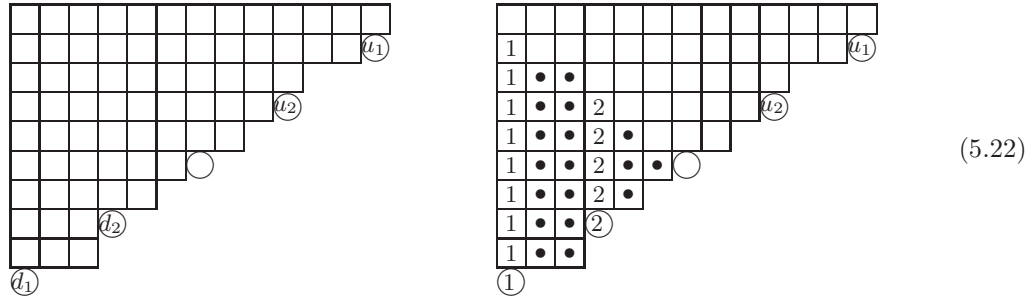
$$v_{(4,2,0;4,1)} = -\frac{(5\alpha + 1)(5\alpha + 3)}{(2\alpha + 1)(\alpha + 1)^2(4\alpha + 3)}, \quad (5.20)$$

which coincides with our data.

One case that is not as trivial as it should be is the case of $v_{(2;5,5)} \in \square_{5,5}$ which, by our choice of normalization, is equal to 1. In this case $\ell = 0$ and so the only non-trivial part is in the first square

$$\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \bullet & \bullet & & & \\ \hline \end{array} \quad (5.21)$$

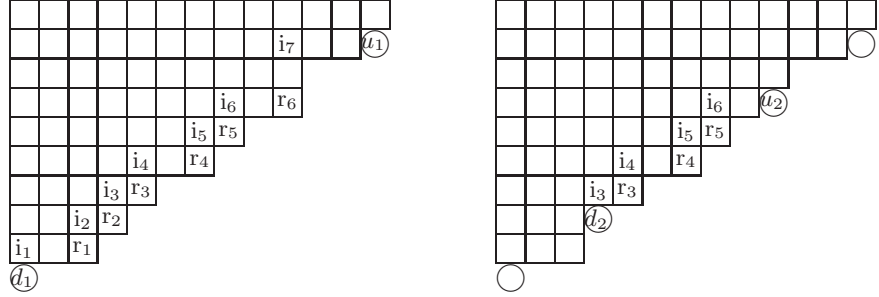
5.2.3 Further illustration of the combinatorics underlying the general formula Here is an example based on a large diagram that will illustrating in more details the combinatorial structure underlying the general formula. Consider $\Lambda = (12, 9, 6, 3, 0; 13, 10, 8, 5, 3) \in \square_{13,11}$. The diagram of Λ , with its complementary pairs of circles (u_1, d_1) and (u_2, d_2) , is displayed on the left:



and

(5.23)

Finally, the interior and removable corners of Λ^{\otimes} between u_1 and d_1 and between u_2 and d_2 are respectively


(5.24)

which correspond respectively to the products

$$\frac{\prod_{j=1}^6 \|d_1, r_j\|}{\prod_{j=1}^7 \|d_1, i_j\|} \quad \text{and} \quad \frac{\prod_{j=3}^5 \|d_2, r_j\|}{\prod_{j=3}^6 \|d_2, i_j\|}. \quad (5.25)$$

The final coefficient is the product of all these expressions.

6 sJack representation of the degenerate Whittaker vector

As mentioned in the introduction, Gaiotto has showed that Whittaker vectors naturally appear in asymptotic limits of the AGT relations [29]. The simplest Whittaker vector is the degenerate one. It is defined as the unique state $|h, g\rangle^{\text{Vir}}$ in the Verma module over the Virasoro algebra, with highest weight state $|h\rangle^{\text{Vir}}$, such that

$$L_0|h, g\rangle^{\text{Vir}} = h|h, g\rangle^{\text{Vir}}, \quad L_1|h, g\rangle^{\text{Vir}} = g|h, g\rangle^{\text{Vir}}, \quad L_2|h, g\rangle^{\text{Vir}} = 0, \quad (6.1)$$

where g is a formal parameter. Note that according to the above definition, $L_n|h, g\rangle^{\text{Vir}} = 0$ for all $n \geq 2$. An explicit representation of $|h, g\rangle^{\text{Vir}}$ in terms of Jacks has been obtained in [48] (whose expression is a degenerate case of the one found in [4], where the Whittaker vector of the deformed Virasoro algebra is related to the Macdonald polynomials).

Here we conjecture a closed-form sJack-expansion for the supersymmetric degenerate Whittaker vector. The latter has been recently studied in the context of the supersymmetric generalization of the asymptotic AGT relations in [6] for the NS sector.

6.1 Basic definitions

Let us concentrate on the NS sector. We say that the bosonic state $|h, g\rangle^{\text{NS}}$, in the Verma module with highest weight state $|h\rangle^{\text{NS}}$, is a degenerate Whittaker vector if it satisfies the following supersymmetric generalization of condition (6.1):

$$L_0|h, g\rangle^{\text{NS}} = h|h, g\rangle^{\text{NS}}, \quad G_{\frac{1}{2}}|h, g\rangle^{\text{NS}} = \phi\sqrt{g}|h, g\rangle^{\text{NS}}, \quad G_{\frac{3}{2}}|h, g\rangle^{\text{NS}} = 0, \quad L_2|h, g\rangle^{\text{NS}} = 0, \quad (6.2)$$

where ϕ is an auxiliary fermionic operator, satisfying $\phi^2 = 1$, introduced only for preserving the coherence of the Bose-Fermi-statistics. Note that the above equations imply

$$L_1|h, g\rangle^{\text{NS}} = g|h, g\rangle^{\text{NS}}, \quad G_{n+\frac{1}{2}}|h, g\rangle^{\text{NS}} = 0, \quad L_n|h, g\rangle^{\text{NS}} = 0, \quad \forall n > 1. \quad (6.3)$$

For solving the system of equations (6.2), it is convenient to expand the Whittaker vector level-by-level as

$$|h, g\rangle^{\text{NS}} = \sum_{k \geq 0} g^k |h\rangle_k^{\text{NS}} + \phi \sum_{k \geq 0} g^{k+\frac{1}{2}} |h\rangle_{k+\frac{1}{2}}^{\text{NS}}. \quad (6.4)$$

where $|h\rangle_k^{\text{NS}}$ denotes a descendant of $|h\rangle^{\text{NS}} \equiv |h\rangle_0^{\text{NS}}$ at level k . Then, the conditions (6.2) can be translated into

$$G_{\frac{1}{2}} |h\rangle_k^{\text{NS}} = |h\rangle_{k-\frac{1}{2}}^{\text{NS}}, \quad G_{\frac{3}{2}} |h\rangle_k^{\text{NS}} = 0, \quad L_2 |h\rangle_k^{\text{NS}} = 0, \quad \forall k \in \frac{\mathbb{N}}{2}. \quad (6.5)$$

6.2 Closed-form formula

We now want to represent the degenerate Whittaker vector at level k as a symmetric superpolynomial:

$$|h\rangle_k^{\text{NS}} \longleftrightarrow W_k = \sum_{\text{level}(\Lambda)=k} w_\Lambda P_\Lambda \quad (6.6)$$

where we recall that $\text{level}(\Lambda)$ denotes the level of the superpartition Λ defined in (3.16). The coefficients are fixed by the following differential equations

$$\mathcal{G}_{\frac{3}{2}} W_k = 0 \quad \mathcal{L}_2 W_k = 0, \quad \forall k > 0, \quad (6.7)$$

the recursion

$$\mathcal{G}_{\frac{1}{2}} W_k = W_{k-\frac{1}{2}}, \quad \forall k > 0, \quad (6.8)$$

and the initial condition $W_0 = 1$.

We have found that the coefficients in (6.6) can be written in a rather simple – albeit nontrivial – way as

$$w_\Lambda = (-1)^{\binom{m}{2}} \alpha^{|\Lambda^\circ|} \left[\prod_{s \in \mathcal{B}\Lambda} \frac{1}{h_\Lambda^\uparrow(s)} \right] \left[\prod_{(i,j) \in \Lambda^\circ} \frac{1}{\bar{\eta} + i - \alpha j} \right] A(\Lambda) \quad (6.9)$$

using, as before, $\bar{\eta} = \sqrt{\alpha} \eta$, and where

$$A(\Lambda) = \prod_{\substack{(i,j) \in \Lambda \equiv \Lambda^* \\ (i,j) \text{ not a removable corner of } \Lambda}} \frac{[2\bar{\eta} + 1 + i + \lambda'_j - \alpha(1 + j + \lambda_i)]_{(i,j) \notin \mathcal{F}\Lambda}}{2\bar{\eta} + 1 + 2i - \alpha(1 + 2j)} \quad (6.10)$$

In the last equation, the symbol $\mathcal{F}\Lambda$ denotes the set of entries $s = (i, j)$ in the diagram of Λ that belong at the same time in a fermionic row and a fermionic column, where a row/column is said to be fermionic if it terminates with a circle. In (6.9), $\mathcal{B}\Lambda$ corresponds to the entries $s = (i, j)$ in the diagram of Λ that do not belong to $\mathcal{F}\Lambda$ [20]. Note that when $m = 0$, although the sJacks reduce to the ordinary Jacks, this is not so for the Whittaker vector (the defining conditions as well as the representation being different).

Let us illustrate the evaluation of $A(\Lambda)$ by displaying some examples using the compact notation $(2\bar{\eta} + a - \alpha b) \equiv (a, b)$. We thus have

$$\begin{array}{c} \square \square \circ \\ \circ \end{array} \rightarrow A = \frac{1}{(3, 3)}; \quad \begin{array}{c} \square \square \\ \square \circ \\ \circ \end{array} \rightarrow A = \frac{(4, 4)}{(3, 3)}; \quad \begin{array}{c} \square \square \square \\ \circ \end{array} \rightarrow A = \frac{(3, 6)}{(3, 3)}; \quad \begin{array}{c} \square \circ \\ \square \circ \\ \square \circ \end{array} \rightarrow A = \frac{(6, 3)}{(3, 3)} \quad (6.11)$$

We see that under conjugation, $(a, b) \rightarrow (b, a)$. Here are few more examples:

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \bigcirc \rightarrow A = \frac{(4, 4)}{(3, 3)(3, 5)}; \quad \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} \rightarrow A = \frac{(3, 6)(4, 5)}{(3, 3)(3, 5)}; \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \bigcirc \rightarrow A = \frac{(4, 4)(4, 5)(5, 4)}{(3, 3)(3, 5)(5, 3)}. \quad (6.12)$$

We exemplify the general formula by displaying the explicit representation of the level 3 component of the degenerate Whittaker vector – modifying slightly the above compact notation as $(p\bar{\eta} + a - \alpha b) \equiv (a, b)_p$; it reads

$$\begin{aligned} W_3 = & \frac{(3, 6)_2}{6(1, 1)_1(1, 2)_1(1, 3)_1(3, 3)_2} P_{(3)}^{(\alpha)} + \frac{\alpha(4, 4)_2}{(2\alpha + 1)(1, 1)_1(1, 2)_1(2, 1)_1(3, 3)_2} P_{(2, 1)}^{(\alpha)} \\ & + \frac{\alpha^2(6, 3)_2}{(\alpha + 1)(\alpha + 2)(1, 1)_1(2, 1)_1(3, 1)_1(3, 3)_2} P_{(1, 1, 1)}^{(\alpha)} - \frac{\alpha}{(1, 1)_1(1, 2)_1(1, 3)_1(2, 1)_1(3, 3)_2} P_{(2, 0)}^{(\alpha)} \\ & - \frac{\alpha^2}{(\alpha + 1)(1, 1)_1(1, 2)_1(2, 1)_1(3, 1)_1(3, 3)_2} P_{(1, 0, 1)}^{(\alpha)} \quad (6.13) \end{aligned}$$

6.3 Norm

The physical relevance of the degenerate Whittaker vector is that the square of its norm is equal to the degenerate limit of the four-point conformal block [6, 29] (see the conclusion for a further discussion of this point). It is thus of interest to compute the norm squared of this vector from the point of view of its polynomial representation. What we want to evaluate is

$$(|k\rangle, |k\rangle)_{c, h}, \quad (6.14)$$

where $|k\rangle \equiv |h\rangle_k^{\text{NS}}$ denotes the level- k degenerate Whittaker vector, and where $(\ , \)_{c, h}$ stands for the usual Hermitian Shapovalov form on the NS highest-weight module. The latter form is non-zero and characterized by the following invariance property:

$$(G_r|\Lambda\rangle, |\Omega\rangle)_{c, h} = (|\Lambda\rangle, G_{-r}|\Omega\rangle)_{c, h} \quad (6.15)$$

for all half-integers r and for all basic states $|\Lambda\rangle, |\Omega\rangle$ of the NS highest-weight module ($|\Lambda\rangle$ is taken to be a shorthand for a state of the form (2.2) using the correspondence (2.22)).

We now want to rephrase the norm squared (6.14) in the language of symmetric superpolynomials. Thanks to eqs (2.8) and (2.19), we already know how to represent any element of the highest-weight module over the NS sector as an element of the space \mathcal{R} of symmetric superpolynomials:

$$G_{-r_1} \cdots G_{-r_p} L_{-n_1} \cdots L_{-n_q} |h\rangle^{\text{NS}} \longleftrightarrow \mathcal{G}_{-r_1} \cdots \mathcal{G}_{-r_p} \mathcal{L}_{-n_1} \cdots \mathcal{L}_{-n_q}(1) = \sum_{\Lambda} u_{\Lambda} P_{\Lambda}^{(\alpha)}, \quad (6.16)$$

where the sum runs over all superpartitions whose level is equal to $\sum_i r_i + \sum_j n_j$ and where the u_{Λ} 's denote complex coefficients depending upon α and η . We recall that these parameters are related to c and h of the module through

$$c = \frac{3}{2} - 12\gamma^2 \quad h = \frac{1}{2}\eta(\eta - 2\gamma), \quad \gamma = \frac{1}{2\sqrt{\alpha}}(\alpha - 1). \quad (6.17)$$

We also know that the space \mathcal{R} is naturally equipped with the scalar product $\langle\langle \ , \ \rangle\rangle_{\alpha}$ defined in (3.11). However, this scalar product is not compatible with the invariance property (6.15). Consider

for instance the operators \mathcal{L}_0 and $\mathcal{G}_{\pm\frac{1}{2}}$ given in eqs (2.32) and (2.33). Then, by letting β be a function of α and by making use of (see [19, eq (5.2)])

$$\langle\langle p_n f | g \rangle\rangle_\beta = \langle\langle f | \beta n \partial_n g \rangle\rangle_\beta \quad \langle\langle \tilde{p}_n f | g \rangle\rangle_\beta = \langle\langle f | \beta \tilde{\partial}_n g \rangle\rangle_\beta, \quad (6.18)$$

one easily checks that $\langle\langle \mathcal{L}_0 f | g \rangle\rangle_\beta = \langle\langle f | \mathcal{L}_0 g \rangle\rangle_\beta$. On the other hand, one also gets $\langle\langle \mathcal{G}_{\frac{1}{2}} f | g \rangle\rangle_\beta \neq \langle\langle f | \mathcal{G}_{-\frac{1}{2}} g \rangle\rangle_\beta$ whenever $\eta \neq \gamma$, which prevents the interpretation of the scalar product as an invariant form.

It is nevertheless possible to relate the scalar product $\langle\langle \cdot | \cdot \rangle\rangle_\alpha$ to the invariant bilinear form $(\cdot, \cdot)_{c,h}$. For this, we first make a slight change of parametrization:

$$\eta = \rho + \gamma \quad \Longrightarrow \quad h = \frac{1}{2}(\rho + \gamma)(\rho - \gamma). \quad (6.19)$$

Let φ denote the complex conjugation. We set

$$\varphi(\bar{\rho}) = -\bar{\rho} \quad \text{and} \quad \varphi(\bar{\gamma}) = \bar{\gamma}, \quad (6.20)$$

where $\bar{\rho} = \sqrt{\alpha}\rho$ (recall that $\bar{\gamma} = \sqrt{\alpha}\gamma$). We thus assume that $\bar{\gamma}$ (resp. $\bar{\rho}$) is real (resp. purely imaginary), which implies that α is real. Note that both c and h , being invariant under the action of φ , must be real. Then we define

$$\langle\langle f | g \rangle\rangle_\beta^\varphi \equiv \langle\langle \varphi(f) | g \rangle\rangle_\beta, \quad (6.21)$$

which is a Hermitian form on \mathcal{R} . Obviously, $\langle\langle \mathcal{L}_0 f | g \rangle\rangle_\beta^\varphi = \langle\langle f | \mathcal{L}_0 g \rangle\rangle_\beta^\varphi$ for any β . Moreover,

$$\langle\langle \mathcal{G}_{\frac{1}{2}} f | g \rangle\rangle_\beta^\varphi - \langle\langle f | \mathcal{G}_{-\frac{1}{2}} g \rangle\rangle_\beta^\varphi = \left(1 + \frac{\beta}{\alpha}\right) (\bar{\rho} - \bar{\gamma}) \langle\langle \tilde{\partial}_0 f | g \rangle\rangle_\beta^\varphi. \quad (6.22)$$

In other words, the invariance criterion requires $\beta = -\alpha$. Direct calculations then show that the latter condition is also sufficient, so that

$$\langle\langle \mathcal{G}_r f | g \rangle\rangle_{-\alpha}^\varphi = \langle\langle f | \mathcal{G}_{-r} g \rangle\rangle_{-\alpha}^\varphi, \quad \forall r \in \mathbb{Z} + \frac{1}{2}, \quad \forall f, g \in \mathcal{R}. \quad (6.23)$$

Let us summarize what we have obtained so far. We are given two isomorphic vector spaces over \mathbb{C} : 1) the highest-weight module over the NS sector characterized by the central charge c and highest weight h ; 2) the space \mathcal{R} of symmetric superpolynomials whose coefficients are complex-valued rational functions of α and $\bar{\rho}$. Both spaces are equipped with a non-zero invariant Hermitian form. Moreover, c and h are real-valued functions of α and $\bar{\rho}$. Now, as is well known (see for instance [31, p.171]), whenever c and h are real, the Hermitian Shapovalov form on the highest-weight module is unique up to a multiplicative complex constant. We can thus conclude that if $|\Lambda\rangle \longleftrightarrow f$ and $|\Omega\rangle \longleftrightarrow g$, then

$$(|\Lambda\rangle, |\Omega\rangle)_{c,h} = z \langle\langle f | g \rangle\rangle_{-\alpha}^\varphi \quad (6.24)$$

for some complex constant z .

We finally return to the evaluation of (6.14). According to (6.6), the degenerate Whittaker $|k\rangle$ is represented in \mathcal{R} as

$$W_k(\alpha, \bar{\rho}) = \sum_{\text{level}(\Lambda)=k} w_\Lambda(\alpha, \bar{\rho}) P_\Lambda^{(\alpha)}, \quad (6.25)$$

where $w_\Lambda(\alpha, \bar{\rho})$ is given by formula (6.9) with $\bar{\eta} = \bar{\rho} + \bar{\gamma}$. Moreover, the equality (6.24) allows us to write

$$(|k\rangle, |k\rangle)_{c,h} = z \langle\langle W_k(\alpha, \bar{\rho}) | W_k(\alpha, \bar{\rho}) \rangle\rangle_{-\alpha}^\varphi = z \langle\langle W_k(\alpha, -\bar{\rho}) | W_k(\alpha, \bar{\rho}) \rangle\rangle_{-\alpha} \quad (6.26)$$

for some numeral constant z . Actually, in order to compare our results with those of [6], it is more convenient to adjust the factor z level by level: we set $z = 4^{-\lfloor k \rfloor}$ if the degenerate Whittaker vector is at level k . Consequently,

$$\boxed{\left(|k\rangle, |k\rangle \right)_{c,h} = 4^{-\lfloor k \rfloor} \sum_{\substack{\Lambda, \Omega \\ \text{level}(\Lambda) = \text{level}(\Omega) = k}} w_\Lambda(\alpha, -\bar{\rho}) w_\Omega(\alpha, \bar{\rho}) \langle\langle P_\Lambda^{(\alpha)} | P_\Omega^{(\alpha)} \rangle\rangle_{-\alpha}}. \quad (6.27)$$

We have checked the concordance of the lhs of (6.27) against all the norms squared $(|k\rangle, |k\rangle)_{c,h}$ calculated in [6] (i.e., up to level $k = 5/2$).

Let us consider a nontrivial example: the evaluation of the norm squared for $k = 3/2$. By setting $\bar{\eta} = \bar{\rho} + \bar{\gamma}$ with $\bar{\gamma} = (\alpha - 1)/2$ in eq (6.9), we find

$$W_{\frac{3}{2}}(\alpha, \bar{\rho}) = \frac{4}{(1 - 3\alpha + 2\bar{\rho})(1 - \alpha + 2\bar{\rho})} P_{(1;)}^{(\alpha)} + \frac{4\alpha}{(\alpha + 1)(2\bar{\rho} - \alpha + 3)(1 - \alpha + 2\bar{\rho})} P_{(0;1)}^{(\alpha)} \quad (6.28)$$

Using the decomposition of the above two sJacks in power sums, which are given in eqs (4.9) and (4.13) (see also Appendix C), we find

$$\langle\langle P_{(0;1)}^{(\alpha)} | P_{(0;1)}^{(\alpha)} \rangle\rangle_{-\alpha} = \alpha(\alpha - 1), \quad \langle\langle P_{(0;1)}^{(\alpha)} | P_{(1;)}^{(\alpha)} \rangle\rangle_{-\alpha} = \frac{2\alpha^2}{(\alpha + 1)}, \quad \langle\langle P_{(1;)}^{(\alpha)} | P_{(1;)}^{(\alpha)} \rangle\rangle_{-\alpha} = -\frac{\alpha^2(\alpha - 1)}{(\alpha + 1)^2}. \quad (6.29)$$

The rhs of (6.27) is thus equal to

$$\frac{-4(-9\alpha^2 + 22\alpha - 9 + 4\bar{\rho}^2)\alpha^2}{[(1 - \alpha)^2 - 4\bar{\rho}^2][(3 - \alpha)^2 - 4\bar{\rho}^2][(1 - 3\alpha)^2 - 4\bar{\rho}^2]} = \frac{c + 3h}{8h(c - 9h + 2ch + 6h^2)} \quad (6.30)$$

which in turn is exactly equal to the expression $\langle\frac{3}{2}|\frac{3}{2}\rangle$ in [6].⁵

7 Conclusion

Jack polynomials made an unexpected entry in CFT as a closed-form polynomial representation of singular vectors [3, 37, 41]. This is a fascinating, but still somewhat mysterious relationship. One of our original motivations for studying the eigenfunctions of the supersymmetric Sutherland model was precisely to generalize this connection at the SCFT level (cf. the concluding remarks in [16]). This goal is achieved here. As plainly indicated in the main text, the supersymmetric case is more tricky since a linear combination of sJacks is required to describe a singular vector. Our most important technical result is the exact determination of the relative coefficients in this representation of a generic NS singular vector.

It should be stressed that this general formula is provable; although its demonstration is a difficult problem, it is purely technical since we have all the ingredients at hand. Indeed, to demonstrate that a given combination of sJacks represents a singular vector, it suffices to show that it is annihilated by $\mathcal{G}_{\frac{1}{2}}$ and $\mathcal{G}_{\frac{3}{2}}$. For this, one needs the explicit action of these operators on the sJacks. Note that instead of $\mathcal{G}_{\frac{3}{2}}$, one can use the expressions of $\mathcal{G}_{-\frac{1}{2}}$ and \mathcal{L}_2 given that their commutation generates the desired $\mathcal{G}_{\frac{3}{2}}$. Since we have derived in [21] the result of the action of $\mathcal{G}_{\frac{1}{2}}$ and $\mathcal{G}_{-\frac{1}{2}}$ on $P_\Lambda^{(\alpha)}$ via a set of Pieri-like rules, it remains to find the action of \mathcal{L}_2 . But \mathcal{L}_2 is essentially obtained from the commutation of ∂_2

⁵Note that the notation used in [6] is slightly different from ours. One goes from the former to the latter as follows: $\Delta \mapsto h$ and $c \mapsto 2c/3$.

and $2D + 3\Delta$ (cf. (3.1) and (3.2)), where the latter acts diagonally on the sJacks and the former can be obtained as (anti)commutators of operators whose action was given in [21].

Our second main result concerns the expression of the degenerate Whittaker vector, in both the NS and R sectors, in terms of a sum of sJacks with all the coefficients determined explicitly (following the argument detailed in the previous paragraph, we believe that this result is also provable with the tools we have). Let us put this contribution in the context of the recent work revolving around the AGT conjecture.

The Jack-CFT correspondence took a dramatic twist recently in works related to the AGT conjecture [2]. Recall that this conjecture gives a relation between the conformal blocks in 2D CFT and 4D $\mathcal{N} = 2$ supersymmetric gauge theories via the instanton part of the Nekrasov partition-function [39]. From the CFT point of view, this provided a totally unexpected approach to the computation of conformal blocks that calls for an intrinsic CFT understanding. This issue was resolved in [1], where by extending the Virasoro algebra by a $u(1)$ factor, a particular orthogonal basis of the extended algebra was obtained having the noteworthy property that the matrix elements of the primary fields in this basis factorize in a very simple way – precisely along the Nekrasov-partition-function pattern. Uncovering this basis essentially amounts to a proof of the $U(2)$ -version of the AGT conjecture on the sphere. Interestingly, this orthogonal basis diagonalizes a set of commuting integrals that turn out to be the quantum Benjamin-Ono conserved quantities. In [27], which was in part motivated by the CFT description of the fractional quantum Hall effect (see for instance [10, 26]), the latter conserved quantities were also directly described as a sum of two coupled Sutherland Hamiltonians.

The above-mentioned basis vectors are specified by two diagrams. A remarkable fact is that whenever one diagram is void, the basis vector is the Jack labeled by the other diagram. This is the starting point for a recursive procedure allowing the construction of the basis in the general case [1].

The supersymmetric generalization of the AGT conjecture was formulated in [6]: the supersymmetry on the CFT side corresponds to the replacement of the \mathbb{R}^4 manifold of the gauge theory by the orbifold $\mathbb{R}^4/\mathbb{Z}_2$. This proposal has been checked in the pure gauge case in [6], and further confirmed in [7], when matter fields are present (for further discussion, see [8, 12, 34]).

In the pure-gauge case, the superconformal blocks are expressed in terms of the norm of the degenerate Whittaker vector. In [6], this norm is calculated order-by-order, up to level 5/2. What we have obtained is a closed-form expression for this norm to all orders (cf. (6.27)). We can thus reformulate the correspondence proposed in [6] as a purely algebraic-combinatorial conjecture:

$$\boxed{Z_{2k} = 4^{-[k]} \sum_{\substack{\Lambda, \Omega \\ \text{level}(\Lambda) = \text{level}(\Omega) = k}} w_{\Lambda}(\alpha, \bar{\rho}) w_{\Omega}(\alpha, -\bar{\rho}) \langle\langle P_{\Lambda}^{(\alpha)} | P_{\Omega}^{(\alpha)} \rangle\rangle_{-\alpha}} \quad (7.1)$$

where Z_{2k} gives the $2k$ instanton-contribution to the partition function Z_{vec}^{-1} .⁶ But the rhs of (7.1) is not tailor-made for a direct comparison with the Nekrasov partition function. Indeed, it is expressed in terms of the scalar product $\langle\langle P_{\Lambda}^{(\alpha)} | P_{\Omega}^{(\alpha)} \rangle\rangle_{-\alpha}$ (see [4, 48] for analogous conclusions in other contexts). Although the sJacks $P_{\Lambda}^{(\alpha)}$ are orthogonal with respect to $\langle\langle | \rangle\rangle_{\alpha}$, this is not so for $\langle\langle | \rangle\rangle_{-\alpha}$. The expression of $\langle\langle P_{\Lambda}^{(\alpha)} | P_{\Omega}^{(\alpha)} \rangle\rangle_{-\alpha}$ being unknown, this might not be a useful result at the end (it certainly raises the issue of trying to find convenient expansions for these quantities). But the potential usefulness of the derived expression for the conformal block is beyond the scope of the present article, whose aim was to demonstrate the relevance of the sJack basis in SCFT.

⁶The exact combinatorial formula for Z_{2N} is given by the coefficient of q^N on the LHS of eq (5.8) of [6] with Z_{vec} given in eq (5.6).

We now expect that the role played by the Jacks in the proof of the AGT conjecture will be performed by the sJacks in the supersymmetric version of the conjecture.⁷

Acknowledgments

This work was supported by the Natural Sciences and Engineering Research Council of Canada; the Fondo Nacional de Desarrollo Científico y Tecnológico de Chile [#1090034 to P.D., #1090016 to L.L.]; and the Comisión Nacional de Investigación Científica y Tecnológica de Chile [Anillo de Investigación ACT56 Lattices and Symmetry]. P.D. wishes to thank Y. Saint-Aubin for useful discussions on superconformal singular vectors and Shapovalov forms.

A Jack polynomials: basis and singular vectors in CFT

The aim of this appendix is to review the relationship between Virasoro singular vectors and Jack polynomials. As a preliminary step, we explain the statement made at the beginning of the introduction, where we claim that the Jacks provide a representation of the Virasoro states in highest-weight modules. Such a representation is to be understood in the context of the free-field representation.

The free-field (or Feigin-Fuchs) representation of the Virasoro algebra [22, 25],

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}, \quad (\text{A.1})$$

refers to the following expression of the Virasoro modes in terms of the free bosonic modes a_n :

$$L_n = -\gamma(n+1)a_n + \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_k a_{n-k} : \quad \text{where} \quad [a_n, a_m] = n\delta_{n+m,0}. \quad (\text{A.2})$$

In the last equation, the symbol $::$ indicates the normal ordering (the largest mode is placed at the right end) and γ stands for a parameter related to the central charge via $c = 1 - 12\gamma^2$.

The highest-weight states in the Fock space with vacuum $|0\rangle$ are given by $e^{\eta \pi_0} |0\rangle \equiv |\eta\rangle$, where π_0 is the zero-mode conjugated to a_0 which means $[a_0, \pi_0] = 1$, so that $a_0|\eta\rangle = \eta|\eta\rangle$. The full module is obtained by acting in all possible ways with a_n , $n < 0$. Now, states in the Fock modules can be represented by symmetric functions. The basic relationships are (see e.g. [3, 14, 41])

$$\begin{aligned} |\eta\rangle &\longleftrightarrow 1 \\ a_0 &\longleftrightarrow \eta \\ a_{-\lambda_1} \cdots a_{-\lambda_\ell} |\eta\rangle &\longleftrightarrow \zeta' p_{\lambda_1} \cdots p_{\lambda_\ell} \quad (\lambda_i \geq \lambda_{i+1} \geq 1) \end{aligned} \quad (\text{A.3})$$

where ζ' is a constant and $p_n = \sum_{i \geq 1} x_i^n$ is the standard n -th power-sum. The precise correspondence between the modes a_n and the polynomial p_n will be taken to be

$$a_{-n} \longleftrightarrow \frac{(-1)^{n-1}}{\sqrt{2\alpha}} p_n \quad \text{and} \quad a_n \longleftrightarrow n(-1)^{n-1} \sqrt{2\alpha} \frac{\partial}{\partial p_n} \quad (\text{A.4})$$

where $n > 0$ and α stands for a free parameter.

The sequence $(\lambda_1, \dots, \lambda_\ell)$ in eq (A.3) represents a partition – denoted λ – of length ℓ . The polynomial $p_{\lambda_1} \cdots p_{\lambda_\ell}$ in (A.3) is the power-sum symmetric function indexed by the partition λ and

⁷This will be most probably first verified in the $c = 3/2$ (or $\alpha = 1$) case, where the sJacks reduce to their Schur versions, generalizing the construction of [5].

written p_λ . The constant $\zeta' \equiv \zeta_\lambda$ is equal to $(-1)^{|\lambda|-\ell}(2\alpha)^{-\ell/2}$. We could thus rewrite the state-polynomial correspondence (A.3) as $a_{-\lambda}|\eta\rangle \longleftrightarrow \zeta_\lambda p_\lambda$ with $a_{-\lambda} = a_{-\lambda_1} \cdots a_{-\lambda_\ell}$. The sum $\sum_i \lambda_i = |\lambda| = n$ is called the degree of the partition.

In the Fock space, a generic term at level n is a linear combination of states $a_{-\lambda}|\eta\rangle$ with $|\lambda| = n$. Such states can thus be represented by linear combinations of the power-sum symmetric functions p_λ of degree n , i.e.,

$$\sum_{|\lambda|=n} c_\lambda a_{-\lambda} |\eta\rangle \longleftrightarrow \sum_{|\lambda|=n} c_\lambda \zeta_\lambda p_\lambda. \quad (\text{A.5})$$

The sum on the rhs (A.5) can be rewritten in terms of any basis of symmetric functions, in particular, the Jack polynomials.

Now, let us turn to the connection with singular vectors. A highest-weight module over the Virasoro algebra is characterized by the highest-weight state $|h\rangle^{\text{Vir}}$ which satisfies the conditions $L_n|h\rangle^{\text{Vir}} = 0$ for $n > 0$ and $L_0|h\rangle^{\text{Vir}} = h|h\rangle^{\text{Vir}}$. Kac modules are special highest-weight modules for which the conformal dimension $h \equiv h_{r,s}$ is related to the central charge c via the following parametrization depending upon a complex number t :

$$c = 13 - 6 \left(t + \frac{1}{t} \right) \quad \text{and} \quad h_{r,s} = \frac{1}{4}(r^2 - 1)t + \frac{1}{4}(s^2 - 1)\frac{1}{t} - \frac{1}{2}(rs - 1), \quad (\text{A.6})$$

where r and s are two positive integers. The remarkable feature of Kac modules is the occurrence of a linear combination of the descendants (generated by the negative Virasoro modes) at level $n = rs$ that behaves like a highest-weight state: this is the singular vector $|\chi_{r,s}\rangle^{\text{Vir}}$. The highest-weight state $|h_{r,s}\rangle^{\text{Vir}}$ characterizing the Kac module with $h_{r,s}$ and c as in (A.6) can be represented by the Fock state $|\eta_{r,s}\rangle$ if we choose

$$h_{r,s} = \frac{\eta_{r,s}^2}{2} - \gamma\eta_{r,s}, \quad \eta_{r,s} = \frac{1}{\sqrt{2t}}((r+1)t - (s+1)) \quad \text{and} \quad \gamma = \eta_{0,0}. \quad (\text{A.7})$$

The remarkable result of Mimachi and Yamada [37] (see also [3, 14, 41]) is that the Jack polynomial indexed by the rectangular partition (r^s) (i.e., the partition with s parts all equal to r) represents, up to a multiplicative constant, the singular vector $|\chi_{r,s}\rangle^{\text{Vir}}$.⁸

$$|\chi_{r,s}\rangle^{\text{Vir}} \longleftrightarrow P_{(r^s)}^{(\alpha)} \quad (\text{when } t = \alpha \text{ and } \eta = \eta_{r,s}). \quad (\text{A.8})$$

B The Ramond sector

B.1 Modules

The Ramond (R) sector corresponds to the superconformal algebra (2.1) in which the subscripts k and l are integers.⁹ Let $|h\rangle^{\text{R}}$ denote the highest-weight vector of conformal dimension h , i.e., $L_n|h\rangle^{\text{R}} = 0 = G_n|h\rangle^{\text{R}}$ for all $n > 0$ and $L_0|h\rangle^{\text{R}} = h|h\rangle^{\text{R}}$. The highest-weight module built from the state $|h\rangle^{\text{R}}$ can be defined as

$$\mathcal{M}^{\text{R}} = \text{span}_{\mathbb{C}}\{ G_{-\Lambda_1} \cdots G_{-\Lambda_m} L_{-\Lambda_{m+1}} \cdots L_{-\Lambda_\ell} |h\rangle^{\text{R}} \} \quad (\text{B.1})$$

⁸The correspondence between the singular vectors for $c = 1$ and the Schur polynomials is a much older result due to Goldstone (see page 322 in [40]) and generalized by Wakimoto, Yamada [45], and Wallach [46].

⁹For a thorough reference on the subject, see [24]. A more representation-theoretical treatment can be found in [33].

where it is understood that

$$m \geq 0, \quad \ell \geq 0, \quad \Lambda_1 > \dots > \Lambda_m \geq 0, \quad \Lambda_{m+1} \geq \dots \geq \Lambda_\ell \geq 0. \quad (\text{B.2})$$

The last equation simply means that $\Lambda = (\Lambda_1, \dots, \Lambda_m; \Lambda_{m+1}, \dots, \Lambda_\ell)$ is a superpartition of fermionic degree m .

The module \mathcal{M}^{R} is somewhat peculiar due to the relation

$$G_0^2 = L_0 - \frac{c}{24}. \quad (\text{B.3})$$

The previous equation indeed implies that $|h\rangle^{\text{R}}$ is supersymmetric only if $h = c/24$. In what follows however, we restrict ourself to the more interesting case for which $h \neq c/24$. Another consequence of (B.3) is that \mathcal{M} can be split into two submodules:

$$\mathcal{M}^{\text{R},\pm} = \text{span}_{\mathbb{C}}\{G_{-\Lambda_1} \cdots G_{-\Lambda_m} L_{-\Lambda_{m+1}} \cdots L_{-\Lambda_\ell} |h\rangle^{\text{R},\pm}\}, \quad (\text{B.4})$$

where $|h\rangle^{\text{R},+} = |h\rangle^{\text{R}}$ and $|h\rangle^{\text{R},-} = G_0 |h\rangle^{\text{R}}$. We say that the elements of $\mathcal{M}^{\text{R},+}$ (resp. $\mathcal{M}^{\text{R},-}$) have a positive (resp. negative) chirality.

An element $|\chi\rangle$ of \mathcal{M} is a singular vector if $G_n |\chi\rangle = 0$ and $L_n |\chi\rangle = 0$ for all $n > 0$. However, using the commutation relations among the G_n 's and L_m 's, it is simple to check that the following two conditions are sufficient:

$$G_1 |\chi\rangle = 0 \quad \text{and} \quad L_1 |\chi\rangle = 0. \quad (\text{B.5})$$

The algebra in the R sector implies moreover that if $|\chi\rangle = \sum_{\Lambda} c_{\Lambda} G_{-\Lambda^a} L_{-\Lambda^s} |h\rangle$ is a singular vector in $\mathcal{M}^{\text{R},+}$ then $|\chi^-\rangle = \sum_{\Lambda} c_{\Lambda} G_{-\Lambda^a} L_{-\Lambda^s} G_0 |h\rangle$ is a singular vector in $\mathcal{M}^{\text{R},-}$, and conversely. In other words, the information about the singular vectors is exactly the same for both submodules. In order to avoid redundancies, from now on, we limit our study to the submodule with positive chirality.

We say that the highest-weight module $\mathcal{M}^{\text{R},+}$ is a Kac module over R (and of positive chirality) whenever the conformal dimension is related to the central charge through the following parametrization:

$$c = \frac{15}{2} - 3 \left(t + \frac{1}{t} \right) \quad \text{and} \quad h_{r,s} = \frac{t}{8}(r^2 - 1) + \frac{1}{8t}(s^2 - 1) - \frac{1}{4}(rs - 1) + \frac{1}{16}. \quad (\text{B.6})$$

where r and s are positive integers such that $r - s$ is odd, while t is a complex number. Such a module is characterized by the presence of two singular vectors at level $rs/2$: a bosonic one, denoted $|\chi_{r,s}\rangle^{\text{R}}$ and its fermionic partner $G_0 |\chi_{r,s}\rangle^{\text{R}}$ (see for instance [24, p. 616]). Except for the cases $r = 1$ and $s = 1$ [47], there are no known closed-form expressions for the singular vector $|\chi_{r,s}\rangle^{\text{R}}$. Below, we give new explicit formulas in the sJack basis.

B.2 Free-field representation

We now turn our attention to the Fock-space representation of the R sector. This requires the introduction of the following superalgebra involving the free-field modes a_n and b_n , with $n \in \mathbb{Z}$, together with the vacuum charge operator π_0 :

$$[a_n, a_m] = n\delta_{n+m,0}, \quad [a_0, \pi_0] = 1, \quad \{b_n, b_m\} = \delta_{n+m,0}, \quad (\text{B.7})$$

all other commutations being zero. Note in particular that $b_0^2 = 1/2$. In other words, $\sqrt{2}b_0$ is an involution in the superalgebra (B.7). Let $|0\rangle$ be the vacuum state, i.e., $a_k |0\rangle = 0 = b_k |0\rangle$ for all $k > 0$. We define a one-parameter family of highest-weight states as $|\eta\rangle \equiv e^{\eta\pi_0} |0\rangle$.

$$a_0 |\eta\rangle = \eta |\eta\rangle, \quad a_n |\eta\rangle = 0 \quad \text{and} \quad b_n |\eta\rangle = 0, \quad \forall n > 0. \quad (\text{B.8})$$

This allows us to introduce the Fock space with highest weight $|\eta\rangle$ over the superalgebra (B.7):

$$\mathcal{F} = \text{span}_{\mathbb{C}}\{ b_{-\Lambda_1} \cdots b_{-\Lambda_m} a_{-\Lambda_{m+1}} \cdots a_{-\Lambda_\ell} |\eta\rangle \} \quad (\text{B.9})$$

where the labeling of the states satisfies (B.2). As one can check, the following equations yield a representation of the R sector on \mathcal{F} :

$$L_n = -\gamma(n+1)a_n + \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_k a_{n-k} : + \frac{1}{4} \sum_{k \in \mathbb{Z}} \left(n - 2k + \frac{1}{2}\right) : b_k b_{n-k} : \quad (\text{B.10})$$

$$G_n = -2\gamma\left(n + \frac{1}{2}\right)b_n + \sum_{k \in \mathbb{Z}} a_k b_{n-k}, \quad (\text{B.11})$$

where the complex parameter γ (the background charge) is related to the central charge via $c = \frac{3}{2} - 12\gamma^2$.

B.3 Correspondence with symmetric superpolynomials

A correspondence between the free modes just described and the differential operators acting on the space \mathcal{R} of symmetric polynomials can be easily established. A convenient choice is

$$a_n \longleftrightarrow \begin{cases} \frac{(-1)^{n-1}}{\sqrt{\alpha}} p_{-n} & n < 0 \\ \eta & n = 0 \\ n\sqrt{\alpha}\partial_n & n > 0 \end{cases} \quad b_n \longleftrightarrow \begin{cases} \frac{(-1)^n}{\sqrt{2}} \tilde{p}_{-n} & n < 0 \\ \frac{1}{\sqrt{2}}(\tilde{p}_0 + \tilde{\partial}_0) & n = 0 \\ (-1)^n \sqrt{2} \tilde{\partial}_n & n > 0 \end{cases}. \quad (\text{B.12})$$

The main differences between the above correspondence and that of the NS sector (2.19) are the absence of α factors in the representation of the b modes and the presence of the zero mode b_0 which is represented by a combination of the fermionic polynomial \tilde{p}_0 and its derivative. Eq (B.12) together with the identification $|\eta\rangle \leftrightarrow 1$ induce the following correspondence between \mathcal{F} and \mathcal{R} :

$$b_{-\Lambda_1} \cdots b_{-\Lambda_m} a_{-\Lambda_{m+1}} \cdots a_{-\Lambda_\ell} |\eta\rangle \longleftrightarrow \zeta_\Lambda p_\Lambda, \quad (\text{B.13})$$

where

$$\zeta_\Lambda = \frac{(-1)^{|\Lambda|-(\ell-m)}}{2^{m/2} \alpha^{(\ell-m)/2}}. \quad (\text{B.14})$$

We stress that, in contradistinction with the NS sector, the level in the Fock space is now exactly equal to the bosonic degree of the superpolynomials.

The free-field representation (B.10) and the correspondence (B.12) yield a representation of the super-Virasoro generators in the R sector as differential operators acting on the space \mathcal{R} of symmetric superpolynomials, for which sample expressions are

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}\eta(\eta - 2\gamma) + \frac{1}{16} + \sum_{m \geq 1} m(p_m \partial_m + \tilde{p}_m \tilde{\partial}_m) \\ \mathcal{G}_0 &= \frac{1}{\sqrt{2}}(\eta - \gamma)(\tilde{p}_0 + \tilde{\partial}_0) - \sqrt{\frac{2}{\alpha}} \sum_{m \geq 1} p_m \tilde{\partial}_m - \sqrt{\frac{\alpha}{2}} \sum_{m \geq 1} m \tilde{p}_m \partial_m \\ \mathcal{G}_1 &= \sqrt{2}(3\gamma - \eta)\tilde{\partial}_1 + \sqrt{\frac{\alpha}{2}}(\tilde{p}_0 + \tilde{\partial}_0)\partial_1 + \sqrt{\frac{\alpha}{2}} \sum_{n \geq 1} n \tilde{p}_n \partial_{n+1} \sqrt{\frac{2}{\alpha}} \sum_{n \geq 1} p_n \tilde{\partial}_{n+1} \\ \mathcal{L}_1 &= \sqrt{\alpha}(\eta - 2\gamma)\partial_1 - \frac{1}{2}(\tilde{p}_0 + \tilde{\partial}_0)\tilde{\partial}_1 - \sum_{n \geq 1} n p_n \partial_{n+1} - \frac{1}{2} \sum_{n \geq 1} (2n+1) \tilde{p}_n \tilde{\partial}_{n+1} \end{aligned} \quad (\text{B.15})$$

The complete differential representation induces the following correspondence between states of the module $\mathcal{M}^{R,+}$ with highest-weight state $|h\rangle^R$ and symmetric polynomials in \mathcal{R} :

$$\sum_{\Lambda} c_{\Lambda} G_{-\Lambda^a} L_{-\Lambda^s} |h\rangle^R \longleftrightarrow \sum_{\Lambda} c_{\Lambda} \mathcal{G}_{-\Lambda^a} \mathcal{L}_{-\Lambda^s}(1) \quad (\text{B.16})$$

where

$$h = \frac{1}{2}\eta(\eta - 2\gamma) + \frac{1}{16}. \quad (\text{B.17})$$

B.4 Singular vectors

As in the NS sector, the singular vectors are captured in a remarkably simple way in the sJack-basis. To apply the correspondence (B.16) to the (r, s) -type Kac module, we must further set

$$t = \alpha, \quad \gamma = \frac{1}{2\sqrt{\alpha}}(\alpha - 1) \quad \text{and} \quad \eta \equiv \eta_{r,s} = \frac{1}{2\sqrt{\alpha}}((r+1)\alpha - (1+s)). \quad (\text{B.18})$$

According to (B.5), the bosonic singular vector $|\chi_{r,s}\rangle^R$ is represented by the superpolynomial $F_{r,s}$ if, and only if,

$$\mathcal{G}_1(F_{r,s}) = 0 \quad \text{and} \quad \mathcal{L}_1(F_{r,s}) = 0 \quad (\text{B.19})$$

where

$$F_{r,s} = \sum_{\Lambda \in \mathcal{P}_{r,s}} c_{\Lambda} P_{\Lambda}^{(\alpha)}, \quad (\text{B.20})$$

and $\mathcal{P}_{r,s}$ denotes the set of superpartitions Λ such that $\deg(\Lambda) = (\frac{rs}{2}|k)$ with $k = 0 \pmod{2}$. The fermionic partner of $|\chi_{r,s}\rangle^R$ is simply represented by $\mathcal{G}_0(F_{r,s})$.

Let us display a few explicit formulas. We choose the normalization in (B.20) to be such that $c_{(r^s/2)} = 1$ and $c_{(r(s-1)/2, r/2)} = 1$ when r is odd or even, respectively. For $r = 1$ and s even, we find

$$|\chi_{1,s}\rangle^R \longleftrightarrow P_{(1^s/2)}^{\alpha} - P_{(1,0;1^{s/2-1})}^{\alpha} \quad (\text{B.21})$$

while its dual expression reads

$$|\chi_{r,1}\rangle^R \longleftrightarrow P_{(\frac{r}{2})}^{\alpha} + \frac{\alpha r}{2} P_{(\frac{r}{2}, 0;)}^{\alpha}. \quad (\text{B.22})$$

The $r = 2$ case is also very simple:

$$|\chi_{2,s}\rangle \longleftrightarrow (\alpha + 1) \sum_{\ell=0}^{(s-1)/2} \prod_{i=\ell+1}^{2\ell+1} \frac{i}{(\alpha + i)} \left(P_{(2^{(s-1)/2-\ell} 1^{2\ell+1})}^{(\alpha)} + \frac{\alpha}{2\ell+1} P_{(1,0;2^{(s-1)/2-\ell} 1^{2\ell})}^{(\alpha)} \right). \quad (\text{B.23})$$

From these examples, we see that the self-complementary rule needs to be slightly modified in the R sector. Indeed, we now have that a superpartition Λ is (r, s) -self-complementary in the R sector if the complement of Λ^* in the rectangle with r columns and s rows corresponds to Λ^* rotated by π . This modified rule has been verified to high degree. For instance, here are the splittings of the rectangles with $(r, s) = (1, 6)$ and $(2, 5)$ (with the boxes of the rotated copy of Λ^* marked with thick frames):

$$(1,6): \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \quad (2,5): \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}. \quad (\text{B.24})$$

In the first case, the only possible value of Λ^* is (1^3) . But this can be associated to two superpartitions: (1^3) and $(1, 0; 1, 1)$. No higher value of the fermionic degree can be realized. Similarly, each $(2, 5)$ -diagram can be realized by two superpartitions, respectively: $(2^2, 1)$ and $(1, 0; 2^2)$, $(2, 1^3)$ and $(1, 0; 2, 1)$ and finally, (1^5) and $(1, 0; 1^3)$. For $r > 2$, there are typically more than two superpartitions associated to each splitting of the rectangle. Consider for instance the case $(r, s) = (3, 4)$; the contributing diagrams and the corresponding superpartitions are

$$\begin{array}{c}
 \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} & \longleftrightarrow & \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} & \longleftrightarrow & \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} & \longleftrightarrow & \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \\
 & & \begin{array}{l} (3, 3) \\ (3, 0; 3) \end{array} & & \begin{array}{l} (3, 2, 1) \\ (1, 0; 3, 2) \\ (2, 0; 3, 1) \\ (2, 1; 3) \\ (3, 0; 2, 1) \\ (3, 1; 2) \\ (3, 2; 1) \\ (3, 2, 1, 0;) \end{array} & & \begin{array}{l} (2, 2, 1, 1) \\ (2, 1; 2, 1) \end{array}
 \end{array} \quad (B.25)$$

B.5 Degenerate Whittaker vector

The NS degenerate Whittaker vector, found to be simple in the sJack basis, also has a simple representation in the R sector. Again, we work within the highest-weight module of positive chirality characterized by the highest weight $|h\rangle^R$. We define the bosonic component of the degenerate Whittaker vector at level k as the state $|h\rangle_k^R$ defined by the conditions [34] :

$$L_1 |h\rangle_k^R = |h\rangle_{k-1}^R, \quad G_1 |h\rangle_k^R = 0, \quad \forall k \geq 1, \quad (B.26)$$

with the initial condition $|h\rangle_0^R = |h\rangle^R$. We have found that these vectors can be represented in the sJack basis as follows:

$$|h\rangle_k^R \longleftrightarrow \tilde{W}_k = 2^k \sum_{\Lambda \in \mathcal{P}_{r,s}} \alpha^{|\Lambda^\oplus|} \tilde{w}_\Lambda P_\Lambda \quad (B.27)$$

where $\mathcal{P}_{r,s}$ denotes the set defined just below eq. (B.20), and

$$\tilde{w}_\Lambda = \left[\prod_{s \in \mathcal{B}\Lambda} \frac{1}{h_\Lambda^\uparrow(s)} \right] \left[\frac{\prod_{(i,j) \in \mathcal{B}\Lambda} [2\bar{\eta} + i + \lambda'_j + \epsilon'_j - \alpha(j + \lambda_i + \epsilon_i)]}{\prod_{(i,j) \in \Lambda} [2\bar{\eta} + 2i - \alpha(2j + 1)] [2\bar{\eta} + 2i + 1 - \alpha 2j]} \right] \quad (B.28)$$

with, as before, $\bar{\eta} = \sqrt{\alpha} \eta$, and where $\epsilon_i = 0$ if the i -th row is fermionic and 1 otherwise and $\epsilon'_j = 0$ if the j -th column is fermionic and 1 otherwise. Note that $\mathcal{B}\Lambda$ was defined after (6.10).

Let us illustrate the evaluation of the second square bracket in (B.28), denoted \tilde{A} below, by displaying some examples using the compact notation $(2\bar{\eta} + a - \alpha b) \equiv (a, b)$. We have

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \circ \rightarrow \tilde{A} = \frac{1}{(3, 2) (2, 3) (2, 5)}, \quad \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \circ \rightarrow \tilde{A} = \frac{1}{(2, 3) (2, 5) (3, 2) (3, 4)}, \quad (B.29)$$

and

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \rightarrow \tilde{A} = \frac{(4, 4) (4, 5) (5, 4) (5, 5)}{(2, 3) (2, 5) (4, 3) (4, 5) (3, 2) (3, 4) (5, 2) (5, 4)}. \quad (B.30)$$

Finally, specializing the general expression of \tilde{W}_k to $k = 2$, we get,

$$\begin{aligned}
 \tilde{W}_2 &= \frac{2(3, 5)}{(2, 3)(2, 5)(3, 2)} P_{(:2)}^{(\alpha)} + \frac{4\alpha(5, 3)}{(\alpha + 1)(3, 2)(2, 3)(5, 2)} P_{(:1,1)}^{(\alpha)} \\
 &+ \frac{4\alpha}{(2, 3)(2, 5)(3, 2)} P_{(2,0)}^{(\alpha)} + \frac{4\alpha^2}{(\alpha + 1)(3, 2)(2, 3)(5, 2)} P_{(1,0;1)}^{(\alpha)}. \quad (B.31)
 \end{aligned}$$

C Tables of Jack superpolynomials

C.1 sJacks in terms of monomials

We are interested in the transition matrices $c_{\Lambda\Omega}$ relating the sJack basis to the supermonomial basis, i.e., $P_\Lambda = \sum_\Omega c_{\Lambda\Omega} m_\Omega$. To make the presentation more compact, we eliminate the parentheses and the commas of the superpartitions, so that, for instance, $(1, 0; 3, 2)$ becomes $10; 32$. In the following tables, the superpartitions labeling the rows are the $P^{(\alpha)}$ -labels and the row-entries are the coefficients of the monomials labeled by the partition specifying the column. For instance, the simplest nontrivial transition matrix and its corresponding explicit formulas read

$$\begin{array}{c|cc} P \backslash m & 1; & 0; 1 \\ \hline 1; & 1 & (\alpha+1)^{-1} \\ 0; 1 & 0 & 1 \end{array} \iff P_{1;}^{(\alpha)} = m_1; + \frac{1}{\alpha+1} m_{0;1}, \quad \text{and} \quad P_{0;1}^{(\alpha)} = m_{0;1}.$$

Below we display all transition matrices for superpartitions of degree $(n|m)$ with $n, m \leq 3$. For $n = 2$, we have

$$\begin{array}{c|cc} P \backslash m & 2 & 11 \\ \hline 2 & 1 & 2(\alpha+1)^{-1} \\ 11 & 0 & 1 \end{array} \quad \begin{array}{c|cc} P \backslash m & 20; & 10; 1 \\ \hline 20; & 1 & (\alpha+1)^{-1} \\ 10; 1 & 0 & 1 \end{array}$$

and

$$\begin{array}{c|cccc} P \backslash m & 2; & 0; 2 & 1; 1 & 0; 11 \\ \hline 2; & 1 & (1+2\alpha)^{-1} & 2(1+2\alpha)^{-1} & 2(\alpha+1)^{-1}(1+2\alpha)^{-1} \\ 0; 2 & 0 & 1 & (\alpha+1)^{-1} & 2(\alpha+1)^{-1} \\ 1; 1 & 0 & 0 & 1 & 2(2+\alpha)^{-1} \\ 0; 11 & 0 & 0 & 0 & 1 \end{array}$$

For $n = 3$, apart from $P_{210;} = m_{210;}$, we have

$$\begin{array}{c|ccc} P \backslash m & 3 & 21 & 111 \\ \hline 3 & 1 & \frac{3}{1+2\alpha} & \frac{6}{(\alpha+1)(1+2\alpha)} \\ 21 & 0 & 1 & \frac{6}{2+\alpha} \\ 111 & 0 & 0 & 1 \end{array} \quad \begin{array}{c|ccccc} P \backslash m & 30; & 21; & 20; 1 & 10; 2 & 10; 11 \\ \hline 30; & 1 & \frac{1}{1+2\alpha} & \frac{2}{1+2\alpha} & \frac{1}{1+2\alpha} & \frac{2}{(\alpha+1)(1+2\alpha)} \\ 21; & 0 & 1 & \frac{1}{\alpha+1} & -\frac{1}{2(\alpha+1)^2} & \frac{1}{(\alpha+1)^2} \\ 20; 1 & 0 & 0 & 1 & \frac{1}{\alpha+1} & \frac{2}{\alpha+1} \\ 10; 2 & 0 & 0 & 0 & 1 & \frac{2+\alpha}{1} \\ 10; 11 & 0 & 0 & 0 & 0 & 1 \end{array}$$

and

$$\begin{array}{c|ccccccc} P \backslash m & 3; & 0; 3 & 2; 1 & 1; 2 & 0; 21 & 1; 11 & 0; 111 \\ \hline 3; & 1 & \frac{1}{1+3\alpha} & \frac{3}{1+3\alpha} & \frac{3(\alpha+1)}{(1+2\alpha)(1+3\alpha)} & \frac{3}{(1+2\alpha)(1+3\alpha)} & \frac{6}{(1+2\alpha)(1+3\alpha)} & \frac{6}{(1+2\alpha)(1+3\alpha)(\alpha+1)} \\ 0; 3 & 0 & 1 & \frac{1}{1+2\alpha} & \frac{1+2\alpha}{1} & \frac{1+2\alpha}{2+\alpha} & \frac{(\alpha+1)(1+2\alpha)}{2\alpha+3} & \frac{(\alpha+1)(1+2\alpha)}{3} \\ 2; 1 & 0 & 0 & 1 & \frac{1}{\alpha+1} & \frac{2(\alpha+1)^2}{1} & \frac{(\alpha+1)^2}{2} & \frac{(\alpha+1)^2}{6} \\ 1; 2 & 0 & 0 & 0 & 1 & \frac{1}{\alpha+1} & \frac{\alpha+1}{2} & \frac{(2+\alpha)(\alpha+1)}{6} \\ 0; 21 & 0 & 0 & 0 & 0 & 1 & \frac{2+\alpha}{2} & \frac{2+\alpha}{3} \\ 1; 11 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{3+\alpha}{1} \\ 0; 111 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

C.2 sJacks in terms of power-sums

We now write down some transition matrices $e_{\Lambda,\Omega}$ relating the sJack basis to that constructed with the power-sums, i.e., $P_\Lambda = \sum_\Omega e_{\Lambda\Omega} p_\Omega$. The simplest nontrivial transition matrix is found at degree $(1|0)$ corresponding to eqs (4.13) and (4.9):

$$\begin{array}{c|cc} P \setminus p & 1; & 0; 1 \\ \hline 1; & \frac{\alpha}{\alpha+1} & \frac{1}{\alpha+1} \\ 0; 1 & -1 & 1 \end{array}$$

For $n = 2$, we have

$$\begin{array}{c|cc} P \setminus p & 2 & 11 \\ \hline 2 & \frac{\alpha}{\alpha+1} & \frac{1}{\alpha+1} \\ 11 & -\frac{1}{2} & \frac{1}{2} \end{array} \quad \begin{array}{c|cc} P \setminus p & 20; & 10; 1 \\ \hline 2, 0; & \frac{\alpha}{\alpha+1} & \frac{1}{\alpha+1} \\ 10; 1 & -1 & 1 \end{array}$$

and

$$\begin{array}{c|cccc} P \setminus p & 2; & 0; 2 & 1; 1 & 0; 11 \\ \hline 2; & 2\alpha^2(\alpha+1)^{-1}(2\alpha+1)^{-1} & \alpha(\alpha+1)^{-1}(2\alpha+1)^{-1} & 2\alpha(\alpha+1)^{-1}(2\alpha+1)^{-1} & (\alpha+1)^{-1}(2\alpha+1)^{-1} \\ 0; 2 & -\alpha(\alpha+1)^{-1} & \alpha(\alpha+1)^{-1} & -(\alpha+1)^{-1} & (\alpha+1)^{-1} \\ 1; 1 & -\alpha(\alpha+2)^{-1} & -(\alpha+2)^{-1} & \alpha(\alpha+2)^{-1} & (\alpha+2)^{-1} \\ 0; 11 & 1 & -2^{-1} & -1 & 2^{-1} \end{array}$$

For $n = 3$, we have the trivial case $P_{210;} = p_{210;}$ and

$$\begin{array}{c|ccc} P \setminus p & 3 & 21 & 111 \\ \hline 3 & \frac{2\alpha^2}{[1,1][2,1]} & \frac{3\alpha}{[1,1][2,1]} & \frac{1}{[1,1][2,1]} \\ 21 & -\frac{\alpha}{[1,2]} & \frac{1}{[1,2]} & \frac{1}{[1,2]} \\ 111 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{array} \quad \begin{array}{c|ccccc} P \setminus p & 30; & 21; & 20; 1 & 10; 2 & 10; 11 \\ \hline 30; & \frac{2\alpha^2}{[1,1][2,1]} & 0 & \frac{2\alpha}{[1,1][2,1]} & \frac{\alpha}{[1,1][2,1]} & \frac{1}{[1,1][2,1]} \\ 21; & -\frac{\alpha}{2[1,1]^2} & \frac{\alpha[2,1]}{2[1,1]^2} & \frac{\alpha}{[1,1]^2} & \frac{1}{2[1,1]} & \frac{1}{2[1,1]^2} \\ 20; 1 & -\frac{\alpha}{[1,1]} & -\frac{\alpha}{[1,1]} & \frac{1}{[1,1]} & 0 & \frac{1}{[1,1]} \\ 10; 2 & -\frac{\alpha}{[1,2]} & 1 & -2 & \frac{[1,1]}{[1,2]} & \frac{1}{[1,2]} \\ 10; 11 & 1 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} \end{array}$$

where we have used the following shorthand notation:

$$[a, b] \equiv a\alpha + b. \quad (\text{C.1})$$

Finally,

$$\begin{array}{c|ccccccc} P \setminus p & 3; & 0; 3 & 2; 1 & 1; 2 & 0; 21 & 1; 11 & 0; 111 \\ \hline 3; & 6\alpha^3 & 2\alpha^2 & 6\alpha^2 & 3\alpha^2 & 3\alpha & 3\alpha & 1 \\ 0; 3 & \frac{1}{[1,1][2,1][3,1]} & \frac{1}{[1,1][2,1][3,1]} & \frac{1}{[1,1][2,1][3,1]} & \frac{1}{[1,1][2,1][3,1]} & \frac{1}{[1,1][2,1][3,1]} & \frac{1}{[1,1][2,1][3,1]} & \frac{1}{[1,1][2,1][3,1]} \\ 2; 1 & \frac{1}{[1,1]^2} & \frac{1}{2[1,1]^2} & \frac{1}{2[1,1]^2} & \frac{1}{2[1,1]^2} & \frac{1}{[1,1]^2} & \frac{1}{[1,1]^2} & \frac{1}{2[1,1]^2} \\ 1; 2 & -\frac{\alpha}{[1,1][1,2]} & -\frac{\alpha}{[1,1][1,2]} & -\frac{\alpha}{[1,1][1,2]} & -\frac{\alpha}{[1,1][1,2]} & -\frac{\alpha}{[1,1][1,2]} & -\frac{\alpha}{[1,1][1,2]} & -\frac{\alpha}{[1,1][1,2]} \\ 0; 21 & \frac{2\alpha}{[1,2]} & \frac{1}{[1,2]} & \frac{1}{[1,2]} & \frac{1}{[1,2]} & \frac{1}{[1,2]} & \frac{1}{[1,2]} & \frac{1}{[1,2]} \\ 1; 11 & \frac{\alpha}{[1,3]} & \frac{1}{[1,3]} & \frac{1}{[1,3]} & \frac{1}{2[1,3]} & \frac{1}{2[1,3]} & \frac{1}{2[1,3]} & \frac{1}{2[1,3]} \\ 0; 111 & -1 & \frac{1}{3} & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{array}$$

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